Indian J. pure appl. Math., 17 (9) : 1154-1163, September 1986

FINITE ELEMENT METHOD IN THE FLOW OF A SECOND-ORDER FLUID BETWEEN TWO POROUS COAXIAL CIRCULAR CYLINDERS

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(*Received* 22 *October* 1985)

The present paper deals with the solution of the steady motion of a second-order fluid through the annulus between two coaxial right circular cylinders with suction and injection by finite element method.

Galerkin's method is used to reduce the linear boundary value problem into the corresponding set of algebraic equations. Gauss-Seidel iterative scheme is used to obtain the solution upto the accuracy of the order 1.0×10^{-8}. The flow of the elastico-viscous fluid is taken as a particular case. The effects of second-order forces in the flow and that of suction and injection are discussed in detail and shown graphically.

1. INTRODUCTION

The study of flows in the presence of porous walls has many applications in the field of engineering, technology and biophysics. The viscous flow past a circular cylinder and like problems are studied by many researchers[1-4]. Datta[5] has solved the problem of steady motion of an idealized visco-elastic liquid through an annulus between coaxial circular cylinders and between two parallel plane boundaries, and flow of a non-Newtonian fluid through an annulus with porous walls. Sharma[6] has extended the problems considered in [4,5] for a second-order fluid.

In the present paper we have investigated the steady motion of an incompressible second-order fluid when it flows through the annulus between two coaxial right circular cylinders with suction and injection[6] by using finite element method. It is assumed that the rate of fluid injection at one of the surface is the same as that of withdrawal at the other, and the inner surface is moving with a constant velocity parallel to itself. The effects of the second-order forces in the flow and that of suction and injection represented by dimensionless parameters T_1, T_2 and R respectively have been discussed and shown graphically.

2. FORMULATION OF THE PROBLEM

Coleman and Noll[7] have suggested following constitutive equation for an incompressible second-order fluid:

$$\tau_{ij} = -p_{ij} + 2\mu_1 d_{ij} + 2\mu_2 e_{ij} + 4\mu_3 d^{\alpha}_{i} d_{\alpha j} \quad \ldots(2.1)$$

where

$$d_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i})$$

and

$$e_{ij} = \tfrac{1}{2}(a_{i,j} + a_{j,i} + 2 v^{m}_{,i} v_{m,j}) \quad \ldots(2.2)$$

p denotes the hydrostatic pressure and τ_{ij} is the stress-tensor; while v_i and a_i denote velocity and acceleration vectors. μ_1, μ_2 and μ_3 are the coefficients of Newtonian-viscosity, elastico-viscosity and cross-viscosity respectively. Equation (2.1) the momentum equation

$$\rho\left(\frac{\partial v^i}{\partial t} + v_{i,j} v^j\right) = \tau^{j}_{i,j} \quad \ldots(2.3)$$

and the equation of continuity for steady motion

$$v^{i}_{,i} = 0 \quad \ldots(2.4)$$

where ρ is the fluid-density and a suffix following a comma denotes covariant differentiation, form the set of governing equations.

Markovitz[8] has shown that 5.46 per cent solution of polyisobutylene in cetane at 30°C behaves as a second-order fluid. The values of the material constants μ_1, μ_2 and μ_3 are calculated by him are 18.5, -0.2 and 1.0 respectively (all expressed in C. G. S. system of units). On this basis the ratio of μ_2 and μ_3 has been taken as -0.2 for the purpose of numerical calculations.

We work with cylindrical coordinates (r, θ, z) and choose the axis of boundaries involved to be along the axis of z. The velocity field for the problem can be chosen in the following form :

$$u = u(r),\ v = 0,\ w = w(r). \quad \ldots(2.5)$$

In order that the velocity field (2.5) is compatible with continuity criterion we must have :

$$\frac{d}{dr}(ur) = 0$$

which gives on integration

$$ur = c, \text{ a constant.} \quad \ldots(2.6)$$

Using eqns. (2.1) and (2.2) and velocity field (2.5) we have the physical components of stress tensor as follows :

$$\tau_{rr} = -p + 2\mu_1 \frac{du}{dr} + 2\mu_2 \left[u \frac{d^2u}{dr^2} + 2\left(\frac{du}{dr}\right)^2 + \left(\frac{dw}{dr}\right)^2 \right] + \mu_3 \left[4\left(\frac{du}{dr}\right)^2 + \left(\frac{dw}{dr}\right)^2 \right] \quad \ldots(2.7)$$

$$\tau_{\theta\theta} = -p + 2\mu_1 \frac{u}{r} + 2\mu_2 \left[\frac{u}{r} \frac{du}{dr} + \frac{u^2}{r^2} \right] + 4\mu_3 \frac{u^2}{r^2} \quad \ldots(2.8)$$

$$\tau_{zz} = -p + \mu_3 \left(\frac{dw}{dr}\right)^2 \quad \ldots(2.9)$$

$$\tau_{rz} = \mu_1 \frac{dw}{dr} + \mu_2 \left[u \frac{d^2w}{dr^2} + \frac{du}{dr} \cdot \frac{dw}{dr} \right] + 2\mu_3 \frac{du}{dr} \cdot \frac{dw}{dr} \quad \ldots(2.10)$$

$$\tau_{r\theta} = \tau_{\theta z} = 0 \quad \ldots(2.11)$$

and equations of motion for steady case are :

$$\rho u \frac{du}{dr} = \frac{d}{du} \tau_{rr} + \frac{1}{r} (\tau_{rr} - \tau_{\theta\theta}) \quad \ldots(2.12)$$

and

$$\rho u \frac{dw}{dr} = \frac{d}{dr} \tau_{rz} + \frac{1}{r} \tau_{rz}. \quad \ldots(2.13)$$

Let us suppose that z-axis is along the common-axis of the cylinders having radii a and b ($a > b$). If the inner cylinder moves with a constant velocity along the direction of the axis and the outer cylinder is fixed, the boundary conditions for the problem are :

$$u = U, \; w = 0 \quad \text{at } r = a$$
$$= \bar{U}, \; w = W \quad \text{at } r = b. \quad \ldots(2.14)$$

From (2.6) and (2.14)

$$aU = ur = \bar{U} b = \text{a constant}$$

or,

$$u = \frac{aU}{r} = \frac{b\bar{U}}{r} = \frac{U}{\eta}$$

where $\eta \left(= \frac{r}{a} \right)$ is dimensionless distance.

Equation (2.13) can be written as :

$$\rho a U \frac{dw}{dr} = \frac{d}{dr} [r \tau_{rz}]$$

or,

$$\tau_{rz} = \frac{\rho a U}{r}(w + D) \qquad \ldots(2.15)$$

where D is a constant. Eliminating τ_{rz}, eqns. (2.10) and (2.15) give :

$$T_1 \frac{d^2w}{d\eta^2} + \frac{1}{\eta}(\eta^2 - T_1 - 2T_2)\frac{dw}{d\eta} - R(w + D) = 0 \qquad \ldots(2.16)$$

where $T_1 = \mu_2 U/\mu_1 a$ and $T_2 = \mu_3 U/\mu_1 a$ are dimensionless elastico-viscous and cross-viscous parameters; $R = Ua\rho/\mu_1$ is dimensionless suction parameter.

The boundary conditions on w can be written as :

$$w = 0 \quad \text{when } \eta = 1, \qquad \ldots(2.17)$$

$$w = W \quad \text{when } \eta = \sigma$$

where σ ($= b/a$) is ratio of the two radii.

3. Solution of the Problem

Let $w^* = \frac{w}{W}$.

After dropping the asteriks, eqns. (2.16) and (2.17) can be written as :

$$T_1 \frac{d^2w}{d\eta^2} + \frac{1}{\eta}\left(\eta^2 - T_1 - 2T_2\right)\frac{dw}{d\eta} - R(w + D) = 0 \qquad \ldots(3.1)$$

and

$$w = 0 \qquad \text{when } \eta = 1, \qquad \ldots(3.2)$$

$$w = 1 \qquad \text{when } \eta = \sigma.$$

Let the solution of eqn. (3.1) be

$$w = \sum_{i=1}^{M} \phi_i(\eta)\, w_i \qquad \ldots(3.3)$$

where $\phi_i(\eta)$ is linear shape function and defined as

$$\phi_i(\eta) = \begin{cases} \dfrac{\eta_{i+1} - \eta}{h} & \text{when } \eta_i \leqslant \eta \leqslant \eta_{i+1} \\ \dfrac{\eta - \eta_{i-1}}{h} & \text{when } \eta_{i-1} \leqslant \eta \leqslant \eta_i \\ 0 & \text{otherwise,} \end{cases}$$

where h is the step-length along η.

Putting the values of first and second derivatives of w and using eqn. (3.3), we get the error term as

$$E(\eta) = T_1 \sum_{i=1}^{M} \phi_i''(\eta)\, w_i + \sum_{i=1}^{M} \frac{1}{\eta_i} \left(\eta_i^2 - T_1 - 2T_2 \right) \phi_i'(\eta)\, w_i - R$$

$$\left(\sum_{i=1}^{M} \phi_i(\eta)\, w_i + D \right). \qquad \ldots(3.4)$$

Applying Galerkin finite element method we get

$$\int_{\sigma}^{1} E(\eta).\ \phi_k\, d\eta = 0,\ K = 2\ (1)\ N \qquad \ldots(3.5)$$

$$\left[-\frac{T_1}{h} - \left(\frac{\eta_{K-1}^2 - T_1 - 2T_2}{2\eta_{K-1}} \right) - \frac{Rh}{6} \right] w_{K-1} + \left(\frac{2T_1}{h} - \frac{2}{3} Rh \right) w_K$$

$$+ \left[-\frac{T_1}{h} + \left(\frac{\eta_{K-1}^2 - T_1 - 2T_2}{2\eta_{K+1}} \right) - \frac{Rh}{6} \right] w_{K+1} = RDh,$$

$$K = 2\ (1)\ N \qquad \ldots(3.6)$$

and boundary conditions (3.2) give

$$w(1) = 0 \quad \text{wnen } \eta = 1,$$

$$w(\sigma) = 1 \quad \text{when } \eta = \sigma. \qquad \ldots(3.7)$$

Equation (3.6) in matrix form can be written as

$$\begin{bmatrix} b & c_1 & & & & \\ a_2 & b & c_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & a_{N-2} & b & c_{N-2} & \\ & & & a_{N-1} & b & \end{bmatrix} \begin{bmatrix} w_2 \\ w_3 \\ \vdots \\ w_{N-1} \\ w_N \end{bmatrix} = \begin{bmatrix} RDh - a_1 w_1 \\ RDh \\ \vdots \\ RDh \\ RDh - c_{N-1} w_{N+1} \end{bmatrix} \qquad \ldots(3.8)$$

where

$$a_i = -\frac{T_1}{h} - \left(\frac{\eta_{i-1}^2 - T_1 - 2T_2}{2\eta_{i-1}} \right) \frac{Rh}{6}$$

$$b = \frac{2T_1}{h} - \frac{2}{3} Rh$$

$$c_i = -\frac{T_1}{h} + \left(\frac{\eta_{i+1}^2 - T_1 - 2T_2}{2\eta_{i+1}}\right) - \frac{Rh}{6},\ i = 1\,(1)\,N-1$$

or,

$$W_K^{(S+1)} = \frac{1}{b}\left(RDh - a_{K-1}\ w_{K-1}^{(S)} - c_{K-1}\ w_{K+1}^{(S)}\right)\ K = 2\,(1)\,N. \quad ...(3.9)$$

Computations are made for $W_K^{(S+1)}$ $(K = 2\,(1)\,N)$ by using the convergence criteria to achieve the desired accuracy (1.0×10^{-8})

Particular Cases :

(i) by putting $T_2 = 0$, we get the results for the elastico-viscous fluid.

(ii) by putting the boundary conditions as

$w = 0$ when $\eta = 1$

$w = 0$ when $\eta \to \infty$.

The results are matched with Sharma *et al.*[9].

4. Conditions for Convergence of the Iterative Process

Equation (3.8) can be written as

$$\begin{aligned} w_2 &= F_1\,(w_1, w_3) \\ w_3 &= F_2\,(w_2, w_4) \\ &\vdots \\ w_N &= F_{N-1}\ (w_{N-1}, w_{N+1}). \end{aligned} \quad ...(4.1)$$

Then if $w_1^{(0)}, w_2^{(0)}, \ldots, w_{N+1}^{(0)}$ be the approximate values of $w_1, w_2 \ldots, w_{N+1}$ improved values are found by the following steps :

$$\begin{aligned} w_2^{(1)} &= F_1\left(w_1^{(0)}, w_3^{(0)}\right) \\ w_3^{(1)} &= F_2\left(w_2^{(0)}, w_4^{(0)}\right) \\ &\vdots \\ w_N^{(1)} &= F_{N-1}\left(w_{N-1}^{(0)}, w_{N+1}^{(0)}\right). \end{aligned} \quad ...(4.2)$$

Subtracting this system from eqn. (4.1), we have

$$w_2 - w_2^{(1)} = F_1\,(w_1, w_3) - F_1\left(w_1^{(0)}, w_3^{(0)}\right)$$

$$w_3 - w_3^{(1)} = F_2(w_2, w_4) - F_2\left(w_2^{(0)}, w_4^{(0)}\right)$$

$$\vdots$$

$$w_N - w_N^{(1)} = F_{N-1}(w_{N-1}, w_{N+1}) - F_{N-1}\left(w_{N-1}^{(0)}, w_{N+1}^{(0)}\right). \qquad \ldots(4.3)$$

Now applying to the right-hand side of the system (4.3) the mean value of theorem we have

$$F_1(w_1, w_3) - F_1\left(w_1^{(0)}, w_3^{(0)}\right) = \left(w_1 - w_1^{(0)}\right)\frac{\partial \bar{F}_1}{\partial w_1} + \left(w_3 - w_3^{(0)}\right)\frac{\partial \bar{F}_1}{\partial w_3}$$

where

$$\frac{\partial \bar{F}_1}{\partial w_1} = \frac{\partial \bar{F}_1\left[w_1^{(0)} + \theta\left(w_1 - w_1^{(0)}\right), w_3^{(0)} + \theta\left(w_3 - w_3^{(0)}\right)\right]}{\partial w_1}, \quad 0 \leqslant \theta \leqslant 1$$

and

$$\frac{\partial \bar{F}_1}{\partial w_2} = \frac{\partial F_1\left[w_1^{(0)} + \theta\left(w_1 - w_1^{(0)}\right), w_3^{(0)} + \theta\left(w_3 + w_3^{(0)}\right)\right]}{\partial w_3}, \quad 0 \leqslant \theta \leqslant 1.$$

In a similar manner we get

$$F_2(w_2, w_4) - F_2\left(w_2^{(0)}, w_4^{(0)}\right) = \left(w_2 - w_2^{(0)}\right)\frac{\partial \bar{F}_2}{\partial w_2} + \left(w_4 - w_4^{(0)}\right) \times \frac{\partial \bar{F}_2}{\partial w_4}$$

$$F_3(w_3, w_5) - F_3\left(w_3^{(0)}, w_5^{(0)}\right) = \left(w_3 - w_3^{(0)}\right)\frac{\partial \bar{F}_3}{\partial w_3} + \left(w_5 - w_5^{(0)}\right)\frac{\partial \bar{F}_3}{\partial w_5},$$

$$\vdots$$

$$F_{N-1}(w_{N-1}, w_{N+1}) - F_{N-1}\left(w_{N-1}^{(0)}, w_{N+1}^{(0)}\right) = \left(w_{N-1} - w_{N-1}^{(0)}\right)\frac{\partial \bar{F}_{N-1}}{\partial w_{N-1}} + \left(w_{N+1} - w_{N+1}^{(0)}\right) \times \frac{\partial F_{N-1}}{\partial w_{N+1}}.$$

Substituting these expressions for the right-hand members of eqn. (4.3), it becomes

$$w_2 - w_2^{(1)} = \left(w_1 - w_1^{(0)}\right)\frac{\partial \bar{F}_1}{\partial w_1} + \left(w_3 - w_3^{(0)}\right)\frac{\partial \bar{F}_1}{\partial w_3}$$

$$w_3 - w_3^{(1)} = \left(w_3 - w_2^{(0)}\right)\frac{\partial \bar{F}_2}{\partial w_2} + \left(w_4 - w_4^{(0)}\right)\frac{\partial \bar{F}_2}{\partial w_4}$$

$$\vdots$$

$$w_N - w_N^{(1)} = \left(w_{N-1} - w_{N-1}^{(0)}\right)\frac{\partial \bar{F}_{N-1}}{\partial w_{N-1}} + \left(w_{N+1} - w_{N+1}^{(0)}\right)\frac{\partial \bar{F}_{N-1}}{\partial w_{N+1}}. \quad \ldots(4.4)$$

Adding these equations and considering only the absolute values of the several quantities, we have

$$\left|w_2 - w_2^{(1)}\right| + \left|w_3 - w_3^{(1)}\right| + \ldots + \left|w_N - w_N^{(1)}\right| \leqslant \left|w_1 - w_1^{(0)}\right|\left|\frac{\partial \bar{F}_1}{\partial w_1}\right| + \left|w_2 - w_2^{(0)}\right|\left|\frac{\partial \bar{F}_2}{\partial w_2}\right| + \left|w_3 - w_3^{(0)}\right|\left|\frac{\partial \bar{F}_1}{\partial w_3}\right| + \left|w_4 - w_4^{(0)}\right|\left|\frac{\partial \bar{F}_2}{\partial w_4}\right| + \ldots \left|w_{N-1} - w_{N-1}^{(0)}\right|\left|\frac{\partial \bar{F}_{N-4}}{\partial w_{N-1}}\right| + \left|w_{N+1} - w_{N+1}^{(0)}\right|\left|\frac{\partial \bar{F}_{N-1}}{\partial w_{N+1}}\right|. \quad \ldots(4,5)$$

Finally, as in Scarborough[10], for convergence we have :

$$\left|\frac{\partial F_1}{\partial w_1}\right| + \left|\frac{\partial F_1}{\partial w_3}\right| < 1$$

$$\left|\frac{\partial F_2}{\partial w_2}\right| + \left|\frac{\partial F_2}{\partial w_4}\right| < 1$$

$$\vdots$$

$$\left|\frac{\partial F_{N-1}}{\partial w_{N-1}}\right| + \left|\frac{\partial F_{N-1}}{\partial w_{N+1}}\right| < 1. \quad \ldots(4.6)$$

Using eqns. (3.8) and (4.1), eqn (4.6) becomes :

$$|b| < |c_1|$$

$$|b| < |a_k| + |c_k|, k = 2(1)N - 2$$

$$|b| < |a_{N-1}|. \quad \ldots(4.7)$$

This is a condition for convergence of the solution.

5. Numerical Values and Discussion .

The numerical computations have been made for $T_1 = -0.01, -0.05, -0.10$; $R = -0.4, -0.8, -1.2, 0.10, 0.15\ 0.20$ and $D = -0.5, -1.0, -1.5, 0.05,$

0.10, 0.15. Figure 1 depicts the behoviour of non dimensional component of velocity when T_1 and R is fixed and the constant D is varying. It is noted that the behaviour of w is unaffected with D. The variation of w with suction parameter R when T_1 is taken as fixed, is illustrated through Figs. 2 and 3 in case of suction and injection on the wall of outer cylinder. In first case, it is observed that the velocity component decreases as the suction parameter increases on the wall of the outer cylinder. In the case of injection, it decreases near the inner cylinder and increases near the outer cylinder with an increase in R. Figure 4 shows the behaviour of w for different values of cross-viscous parameter T_2 when suction parameter is constant. It is found that the velocity component decreases with an increase in T_1 for suction as well as injection.

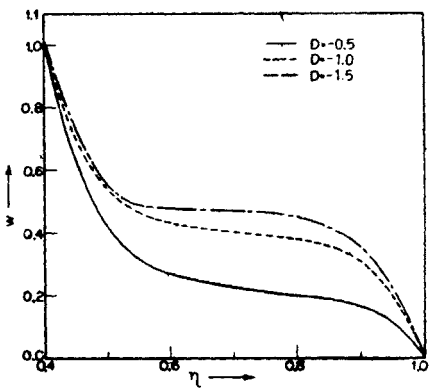

FIG. 1. Variation of Axial Velocity with D for fixed T_1 ($= -0.05$) and R ($= -1.2$).

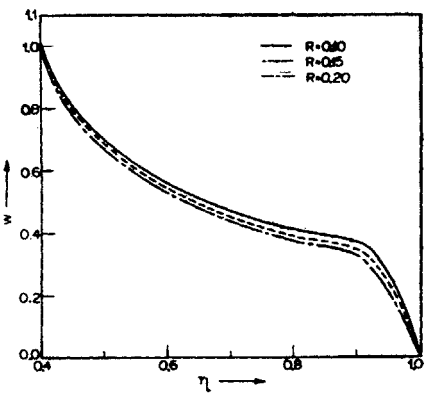

FIG. 2. Variation of Axial Velocity with R (For suction case) for fixed T_1 ($= -0.01$) and D ($= 0.05$).

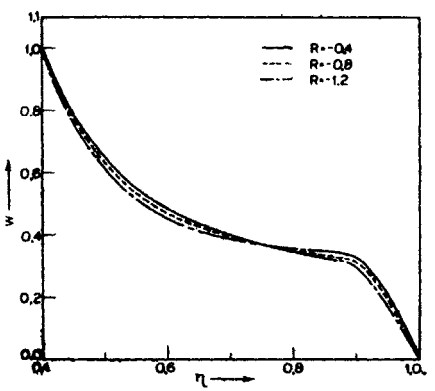

FIG. 3. Variation of Axial Velocity with R (For injection case) for fixed T_1($= -0.01$)and D ($= -0.5$).

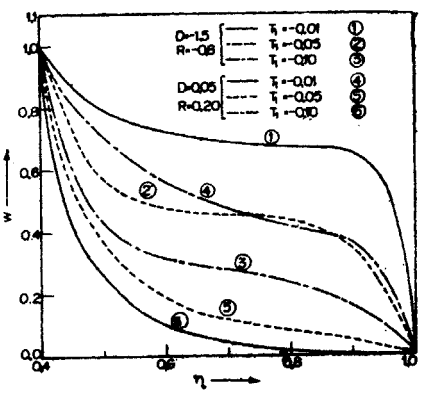

FIG. 4. Variation of Axial Velocity with T_1 (For suction & injection cases for fixed R & D.

ACKNOWLEDGEMENT

The authors are thankful to Dr R. C. Mittal, Deptt. of Mathematics, University of Roorkee, Roorkee for suggestions. One of them (MK) is thankful to the UGC for providing financial assistance in the form of Junior Research Fellowship.

REFERENCES

1. F. W. Meredith, and A. A. Griffith, *A. R. C.* (1955) 3501.
2. J. M. Key, *A.R. C.* (1948), 2628
3. H. G. Lew, *J. Aero. Sci.* **23** (1958), 276.
4. S. P. Misra, *Ganita*, **14** (1963), 99.
5. S. K. Datta, *J. Phys. Soc. Japan* **16** (1966), 794.
6. H. G. Sharma, *J. Phys. Soc. Japan* **21** (1963), 173.
7. B. D. Coleman, and W. Noll *Arch. Ration Mech. Anal.* **6** (1960) 355.
8. H. Markovitz, Refer to A. C. Srivastava, *J. Fluid Mech.* **17** (1963), 171.
9. H. G. Sharma, Manoj Kumar, and Uma Arora, Numerical solution of steady motion of a second-order fluid past a porous circular cylinder under communication. *Ind. Jour Tech.*
10. J. B. Scarborough, *Numerical Mathematical Analysis* Oxford and IBH Publishing Company, 1966, 301.