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IMPLICIT NUMERICAL SOLUTION OF UNSTEADY EULER EQUATIONS FOR TRANSONIC FLOWS

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In the present paper, an implicit method described which makes use of the differential form of the unsteady Euler equations transformed to a rectangular computational domain. This technique entails a substantial reduction in computation time while, at the same time, retaining second-order accuracy. The method is applied to the case of transonic flow past a 6 per cent symmetric circular arc airfoil and a computer program developed. The results from the software for surface pressure distribution on the airfoil have exhibited excellent agreement with the reported experimental results due to Knechtel for a 0.908 Mach flow, as also computed pressure values through an explicit scheme.

NOMENCLATURE

A, B = Jacobian of F, G

c = Speed of sound

D = Diagonal matrix

e = Total energy per unit volume

F, G = Vector flux (see eqn. (1))

I = Identity matrix

J = Jacobian of transformation

n = Superscript representing advance of time

p = Pressure

R = Used to define equation of body surface

R' = See eqn. (4)

S = Diagonalisation matrix

T = Temperature

t = Time

U = Dependent variable $(\rho, \rho u, \rho v, e)^T$

u, v = Velocity components in Cartesian frame

x, y = Cartesian coordinates

Λ = Eigenvalue matrix

ρ = Density

ξ, η = Transformed spatial coordinates

$\Delta t, \Delta x$ = Elemental increase in t, x, ...

Subscript

i, j = i, j mesh points in ξ, η direction

$+$ = Forward difference

$-$ = Backward difference

Superscript

$-$ = Predicted value

$'$ = Transformed coordinate.

1. Introduction

Unsteady Euler equations have the advantage of being applicable in cases of supersonic, subsonic and mixed flows. Even in cases where other simplified flow models, such as steady or potential equations, are valid and could be solved without much difficulty, the simulation of the actual flow from the initial conditions is of relevance not only because it is of considerable theorectical interest, but from the point of view of the insight it provides into the flow development. The solutions of these equations can be obtained by using the explicit time-marching method[2,4,6,7]. Madhavan and Swaminathan[2,6,7]. A major disadvantage of such methods, which are quite accurate, is, however, the stringent restriction on the time-step size and the consequent long computation time necessary to obtain convergence.

MacCormack's[4] implicit predictor-correct numerical scheme, which is second-order accurate, is easily applicable and involves a substantial reduction of computation time by relaxing the restriction on the increment in time, may be made use of to overcome the above difficulty. The method involves only the inversion of bidiagonal matrices instead of tridiagonal ones as is normally required in implicit schemes. This technique is extended in this paper to the case of unsteady Euler equations transformed into a rectangular computational domain from the physical domain of any two-dimensional geometry.

The validation of the algorithm and the associated computer program was carried out by comparing the corresponding results obtained through the software with Knechtel's experimental data for surface pressure distribution for transonic flow past a 6 per cent symmetric circular arc airfoil[1]. The computation time reduces to a small fraction (approximately 30%) of that necessary in cases where the explicit scheme is applied to the integral form of the unsteady Euler equations.

2. Governing Equations and Transformation

The two-dimensional unsteady Euler equations in conservation form are given by

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \qquad \ldots(1)$$

where

$$U = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ e \end{bmatrix}$$

$$F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (e + p)u \end{bmatrix}$$

$$G = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (e+p)\, v \end{bmatrix}. \qquad \ldots(2)$$

The numerical integration of these equations in the physical space is difficult owing to the fact that the grid will not be rectangular and consequently a transformation is made use of. It $y = R(x)$ represents the equation of the body surface (eg. airfoil), the transformation $\xi = x$ and $\eta = y - R(x)$ transforms the computational domain into a rectangular one, while eqn. (1) assumes the form

$$\frac{\partial U'}{\partial t} + \frac{\partial F'}{\partial \xi} + \frac{\partial G'}{\partial \eta} = 0. \qquad \ldots(3)$$

Considering the general transformation $\xi = x$ and $\eta = \eta(x, y)$; we have

$$U' = U/J, \; F' = F/J$$

$$G' = \left(\frac{\partial \eta}{\partial x} F + \frac{\partial \eta}{\partial y} G \right) \Big/ J, \; J = \frac{\partial (\xi, \eta)}{\partial (x, y)}.$$

In the present case,

$$U' = U, \; F' = F$$

$$G' = G - R'(x)\, F. \qquad \ldots(4)$$

For a 6 per cent symmetric circular arc airfoil of chord length unity, the equation of the body surface, referred to rectangular coordinates with the x-axis along the chord length and origin at one end of the chord, is given by

$$x^2 + y^2 - x + 8.3033\, y = 0$$

and consequently the transformed coordinate $\eta = \eta(x, y)$ is obtained as

$$\eta = y - R(x)$$

$$= y - \{ - 8.3033 + \sqrt{8.3033^2 - 4x(x-1)}\}/2.$$

3. The Implicit Numerical Scheme

Differentiating eqn. (3) partially w. r. t. τ, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial t}\right) = -\frac{\partial}{\partial \xi}\left(A\,\frac{\partial U}{\partial t}\right) - \frac{\partial}{\partial \eta}\left(B\,\frac{\partial U}{\partial t}\right) \qquad \ldots(5)$$

where A and B are, respectively, the Jacobians of F and G' w.r.t. U. Implicitly approximating eqn. (5), we have

$$\frac{\frac{\partial U^{n+1}}{\partial t} - \frac{\partial U^n}{\partial t}}{\Delta t} = -\frac{\partial}{\partial \xi}\left(A\,\frac{\partial U}{\partial t}\right)^{n+1} - \frac{\partial}{\partial \eta}\left(B\,\frac{\partial U}{\partial t}\right)^{n+1}$$

or

$$\left(I + \Delta t\,\frac{\partial}{\partial \xi} A. + \Delta t\,\frac{\partial}{\partial \eta} B.\right)\frac{\partial U^{n+1}}{\partial t} = \frac{\partial U^n}{\partial t}. \qquad \ldots(6)$$

Defining

$$\delta U^{n+1} = \frac{\partial U^{n+1}}{\partial t}\,\Delta t$$

and

$$\Delta U^n = \frac{\partial U^n}{\partial t}\,\Delta t$$

$$\left(I + \Delta t\,\frac{\partial}{\partial \xi} A. + \Delta t\,\frac{\partial}{\partial \eta} B.\right)\delta U_{ij}^{n+1} = \Delta U_{ij}^n. \qquad \ldots(7)$$

Now an implicit predictor-corrector scheme may be defined for the numerical integration of the transformed eqns. (3) as follows :

Predictor :

$$\Delta U_{ij}^n = -\Delta t\left(\frac{D_+}{\Delta \xi} F_{ij}^{\prime n} + \frac{D_+}{\Delta \eta} G_{ij}^{\prime n}\right)$$

$$\left(I - \Delta t \ \frac{D_+}{\Delta \xi} \mid A \mid .\right) \left(I - \Delta t \ \frac{D_-}{\Delta \eta} \mid B \mid . \right) \overline{\delta U_{ij}^{n+1}} = \Delta U_{ij}^n$$

$$\overline{U_{ij}^{n+1}} = U_{ij}^n + \delta \overline{U_{ij}^{n+1}}.$$

Corrector :

$$\Delta \overline{U_{ij}^{n+1}} = - \Delta t \left(\frac{D_-}{\Delta \xi} \overline{F_{ij}^{n+1}} + \frac{D_-}{\Delta \eta} \overline{G_{ij}^{n+1}} \right)$$

$$\left(I + \Delta t \ \frac{D_-}{\Delta \xi} \mid A \mid . \right) \left(I + \Delta t \ \frac{D_-}{\Delta \eta} \mid B \mid . \right) \delta U_{ij}^{n+1} = \overline{\Delta U_{ij}^{n+1}}$$

$$U_{ij}^{n+1} = \frac{1}{2} \left[U_{ij}^n + \overline{U_{ij}^{n+1}} + \delta U_{ij}^{n+1} \right]. \qquad \ldots(8)$$

The matrices $\mid A \mid$ and $\mid B \mid$ have positive eigenvalues and are related to Jacobians A and B. Let S_ξ, S_η and their inverses denote the matrices that diagonalise A and B. Then, with perfect gas relations, A and B may be arranged as

$$A = S_\xi^{-1} \Lambda_A S_\xi \qquad \ldots(9)$$

and

$$B = S_\eta^{-1} \Lambda_B S_\eta$$

where

$$S_\xi = \begin{bmatrix} 1 - \frac{\alpha\beta}{C^2} & \frac{u\beta}{C^2} & \frac{v\beta}{C^2} & -\frac{\beta}{C^2} \\ -uc + \alpha\beta & C - u\beta & -v\beta & \beta \\ -\frac{v}{\rho} & 0 & \frac{1}{\rho} & 0 \\ uc + \alpha\beta & -c - u\beta & -v\beta & \beta \end{bmatrix}$$

$$S_\eta = \begin{bmatrix} 1 - \frac{\alpha\beta}{C^2} & \frac{u\beta}{C^2} & \frac{v\beta}{C^2} & -\frac{\beta}{C^2} \\ \frac{\eta_x v - \eta_y u}{\rho\alpha_2} & \frac{\eta_y}{\rho\alpha_2} & -\frac{\eta_x}{\rho\alpha_2} & 0 \\ \beta_1 \left(\alpha\beta - c\frac{\alpha_1}{\alpha_2}\right) & \beta_1 \left(\frac{\eta_x}{\alpha_2} c - u\beta\right) & \beta_1 \left(\frac{\eta_y}{\alpha_2} c - v\beta\right) & \beta_1\beta \\ \beta_1 \left(\alpha\beta + c\frac{\alpha_1}{\alpha_2}\right) & -\beta_1 \left(\frac{\eta_x}{\alpha_2} c + u\beta\right) & -\beta_1 \left(\frac{\eta_y}{\alpha_2} c + v\beta\right) & \beta_1\beta \end{bmatrix}$$

$$\Lambda_A = \begin{bmatrix} u & 0 & 0 & 0 \\ 0 & u+c & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u-c \end{bmatrix}$$

$$\Lambda_B = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_1 + c\alpha_2 & 0 \\ 0 & 0 & 0 & \alpha_1 - c\alpha_2 \end{bmatrix}$$

$$\alpha_1 = \eta_x u + \eta_y v, \quad \alpha_2 = \left(\eta_x^2 + \eta_y^2\right)^{1/2}$$

$$\alpha = \tfrac{1}{2}(u^2 + v^2), \quad \beta = \gamma - 1, \quad \beta_1 = 1/\sqrt{2}\rho c. \qquad \ldots(10)$$

The matrices $|A|$ and $|B|$ are now defined by

$$|A| = S_\xi^{-1} D_A S_\xi$$

and

$$|B| = S_\eta^{-1} D_B S_\eta \qquad \ldots(11)$$

where

$$D_A = \begin{bmatrix} \lambda_{A_1} & 0 & 0 & 0 \\ 0 & \lambda_{A_2} & 0 & 0 \\ 0 & 0 & \lambda_{A_3} & 0 \\ 0 & 0 & 0 & \lambda_{A_4} \end{bmatrix}$$

$$D_B = \begin{bmatrix} \lambda_{B_1} & 0 & 0 & 0 \\ 0 & \lambda_{B_2} & 0 & 0 \\ 0 & 0 & \lambda_{B_3} & 0 \\ 0 & 0 & 0 & \lambda_{B_4} \end{bmatrix} \qquad \ldots(12)$$

$$\lambda_{A_1} = \lambda_{A_3} = \max\left\{ |u| - \frac{1}{2}\frac{\Delta\xi}{\Delta t}, 0.0 \right\}$$

$$\lambda_{A_2} = \max\left\{ |u + c| - \frac{1}{2}\frac{\Delta\xi}{\Delta t}, 0.0 \right\}$$

$$\lambda_{A_4} = \max \left\{ | u - c | - \frac{1}{2} \frac{\Delta \xi}{\Delta t}, 0.0 \right\}$$

$$\lambda_{B_1} = \lambda_{B_2} = \max \left\{ | \alpha_1 | - \frac{1}{2} \frac{\Delta \eta}{\Delta t}, 0.0 \right\}$$

$$\lambda_{B_3} \max \left\{ | \alpha_1 + c\alpha_2 | - \frac{1}{2} \frac{\Delta \eta}{\Delta t}, 0.0 \right\}$$

$$\lambda_{B_4} = \max \left\{ | \alpha_1 - c\alpha_2 | - \frac{1}{2} \frac{\Delta \eta}{\Delta t}, 0.0 \right\}. \quad \ldots(13)$$

For regions of the flow in which Δt satisfies the explicit stability criterion, all λ_A and λ_B vanish and the set of difference equations (8) just reduce to corresponding explicit method.

4. Comruter Program and Results

A computer program was developed in FORTRAN IV language for carrying out the above computations and was made operational on the CDC CYBER170/730 Computer of VSSC. The experimental results used for comparison were those due to Knechtel for Mach 0.908 transonic flow over 6 per cent symmetric circular arc airfoil[1]. The upstream, downstream and top mesh boundary conditions were held fixed at uniform freestream flow in the expectation that the flow near the airfoil will converge before the disturbances reach those points which are held at 10 chord lengths away. The surface pressure on the airfoil was taken to be equal to that at the mesh point immediately above it.

A 48 × 32 mesh system was used in the program. Along the x-direction, a two-mesh system of a stretched mesh (of 10 points) fore and aft of the airfoil and a uniform mesh of 28′ points on the airfoil was made use of. Along the y-direction, a stretched mesh was applied.

The data processing rate was about 8 secs. for iteration, i.e. 2.5×10^{-3} secs. per grid point including prediction and correction, that is, on an average, about 60 per cent more than that for the explicit scheme. Thus, when a CFL number of 5 was used, the implicit scheme took around one-third of the time for the explicit method. CFL numbers up to about 30 could be used without loss of accuracy. A maximum number of 64 iterations were required to obtain convergence when the difference in consecutive flow parameter values becomes less than about 0.1 per cent (Shang and Hankey[5]) and this took around 4.0 minutes on the computer. The corresponding time for the explicit scheme applied to an integral from of the unsteady Euler equations for the same problem was approximately 11.0 min.

Figure 3 presents the computed surface pressure values on the airfoil for Mach 0.908 flow together with the corresponding experimental values due to Knechtel and,

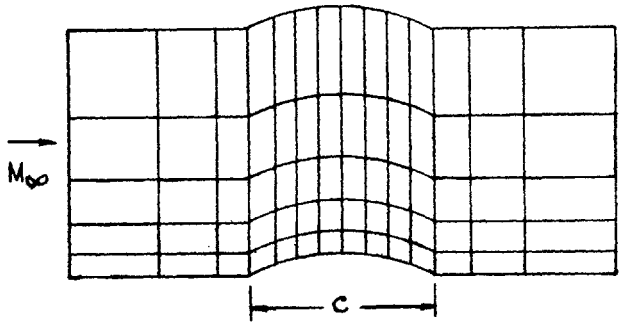

FIG. 1. Mesh about typical circular *ARC* airfoil.

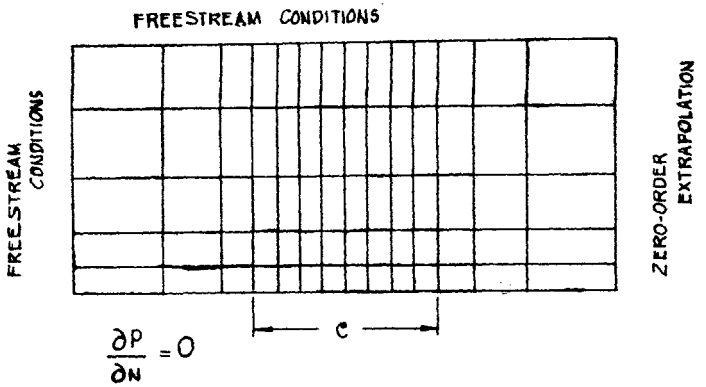

FIG. 2. Mesh system and boundary conditions in transformed computational plane.

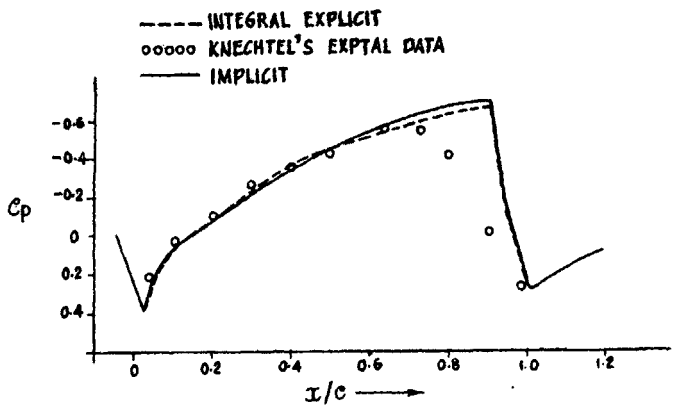

FIG. 3. Comparison of pressure distribution for circular *ARC* airfoil ($M_\infty = 0.908$)

as can be readily observed, the comparison is quite good. The dotted line indicates surface pressure values computed by the corresponding explicit scheme. Figure 4 and

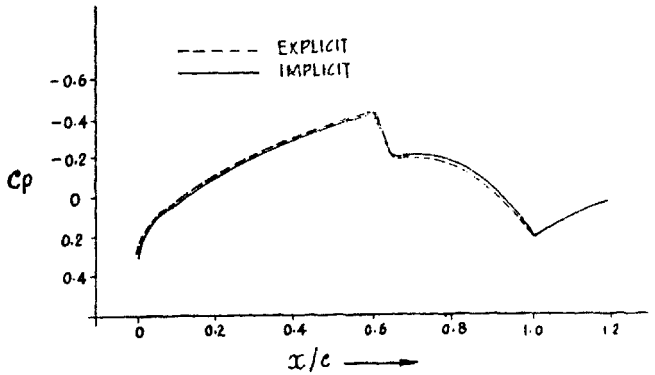

FIG. 4. Pressure distribution for circular *ARC* airfoil ($M_\infty = 0.8$).

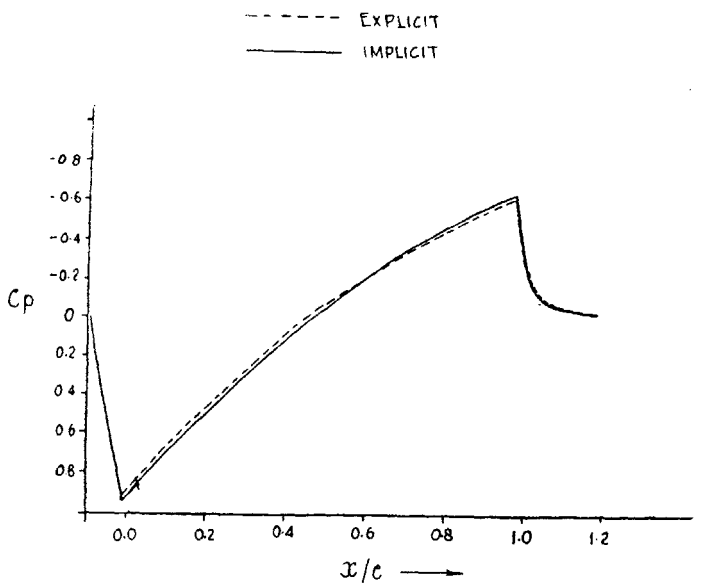

FIG. 5. Pressure Distribution for circular *ARC* airfoil ($M_\infty = 1.2$).

5, respectively, exhibit the computed surface pressure for Mach 0.8 and Mach 1.2 flows.

5. CONCLUSION

An implicit numerical method for solving unsteady Euler equations transformed to a rectangular computational domain is presented in this paper. The method was applied for studying transonic flow past a 6 per cent symmetric circular are airfoil. and a computer program developed for the purpose. The method entails a substantial

reduction in computation time as compared to the corresponding explicit scheme, while retaining second-order accuracy. A comparison of the results for surface pressure distribution obtained through the software for a 0.908 Mach transonic flow past circular arc airfoil with reported experimental values was quite good. Work is presently under way on solving Navier-Stokes equations using an analogous technique for the transonic shock wave boundary layer interaction problem.

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