Indian J. pure appl. Math., 17 (10) : 1175-1186, October 1986

SOME IDENTITIES INVOLVING THE RIEMANN ZETA FUNCTION II

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(*Received* 12 *September* 1985)

By introducing a convolution on the space of certain complex valued sequences, we prove various identities of the following type :

$$4 \sum_{a+b+c=n} \zeta(2a)\,\zeta(2b)\,\zeta(2c) = (n+1)(2n+1)\,\zeta(2n) - 6\zeta(2)\,\zeta(2n-2)$$

where $n \geqslant 3$ is an integer, $\zeta(s)$ denotes the Riemann Zeta function and the sum extends over all ordered triples (a, b, c) of positive integers satisfying $a + b + c = n$.

1. INTRODUCTION

Various identities involving the Riemann Zeta function have been found in the past by many authors including Euler, Ramanujan, Nielsen, Titchmarsh and Eastermann. As a representative of these, we mention the following often rediscovered result

$$\sum_{k=1}^{n-1} \zeta(2k)\,\zeta(2n-2k) = (n - \tfrac{1}{2})\,\zeta(2n),\ n \geqslant 2 \qquad \ldots(1.1)$$

where ζ denotes the Riemann Zeta function defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $s > 1$.

In view of Euler's classic formula (cf. Apostol[1], Theorem 12.17)

$$\zeta(2n) = (-1)^{n-1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!},\ n \geqslant 1 \qquad \ldots(1.2)$$

the identity (1.1) is equivalent to

$$\sum_{k=1}^{n-1} \binom{2n}{2k} B_{2k} B_{2n-2k} = -(2n+1) B_{2n},\ n \geqslant 2. \qquad \ldots(1.3)$$

Here the numbers B_n are Bernoulli numbers and can be defined as the coefficients in the power series

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n,\ |z| < 2\pi.$$

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The identity (1.1) (or its equivalent (1.3)) apparently goes back to Euler (cf. Nielsen[8], p. 49 and footnotes on p. 47) and for various proofs, we refer to S. Ramanujan[10], p. 2, Nielsen[8], p. 49, Underwood[16], Tornheim[15], p. 308, (Williams[17], Theorem 1), Sivaramasarma[13], (Chapter III), Georghiou and Philippou[3] and Sitaramachandrarao[11] (Theorem 3.3). For some related results we refer to Nielsen[8] (pp. 37-51), Titchmarsh[14] (pp. 10-11), Estermann[4], Rademacher[9] (p. 124), Kölbig[6] and Sitaramachandrarao[11]. The methods of these authors are varied—partial fraction decomposition of rational functions, Laurent expansions of meromorphic functions, rearrangement of absolutely convergent double series, Fourier expansions of Eisenstein series and Apostol's extension of a transformation formula due to Lehmer and Newman[2],) (p. 111, Exercise 6:) Sitaramachandrarao[11,12].

The object of this paper is to make a systematic study of identities of the type (1.1) by a method which goes back at least to Ramanujan[10] and Nielsen[8]. In Section 2, we introduce a convolution of complex valued sequences and study some of its properties. In Section 3, we prove some preliminary results and in Section 4 various identities of the type (1.1); the following is typical

$$4 \sum_{a+b+c=n} \zeta(2a)\,\zeta(2b)\,\zeta(2c) = (n+1)(2n+1)\,\zeta(2n) - 6\,\zeta(2)\,\zeta(2n-2)$$

where $n \geqslant 3$ is an integer and the sum extends over all ordered triples (a, b, c) of positive integers satisfying $a + b + c = n$. In Section 5, we give simple proofs of results due to Estermann[4], Williams[17] and prove some more identities.

We note that our results of Sections 3 through 5 (many of which are believed to be new) combined with Euler's formula (1.2) yield various identities involving the Bernoulli numbers. Though these results could also be viewed as recurrence formulae for B_{2n} (or $\zeta(2n)$), we remark that recurrence formulae with large gaps, and hence more convenient for computational purposes, already exist in the literature and we refer to the very interesting work of Lehmer[7].

2. A Convolution

Let E (respectively C) denote the set of all even positive integers (respectively complex numbers). Further let F denote the class of all functions from E to C. For f, $g \in F$, we define

$$(f + g)(2n) = f(2n) + g(2n) \text{ and}$$

$$(f * g)(2n) = \sum_{k=1}^{n-1} f(2k)\, g(2n - 2k) \text{ for } n \geqslant 1$$

with the convention that an empty sum equals zero. Clearly * is commutative, associative and distributes over $+$.

Throughout the paper, we write $\bar{f}(2n)$ for $(2n - 1) f(2n)$.

The following result is basic to our work.

Theorem 2.1—Let $f, g \in F$. Then

$$\overline{f * g} = f * g + f * \bar{g} + \bar{f} * g.$$

PROOF : We have

$$(\overline{f * g})(2n) = (2n-1) \sum_{\substack{a+b=n \\ a \geqslant 1,\ b \geqslant 1}} f(2a)\, g(2b)$$

$$= \sum_{a+b=n} (2a + 2b - 1)\, f(2a)\, g(2b)$$

$$= \sum_{a+b=n} \{(2a-1) + (2b-1) + 1\} f(2a)\, g(2b)$$

$$= \{(\bar{f} * g) + (f * \bar{g}) + (f * g)\}(2n)$$

and the theorem follows.

Corollary 2.1—Let $f \in F$. Then

$$(f * \bar{f})(2n) = (n-1)(f * f)(2n).$$

PROOF : On taking $f = g$ in Theorem 2.1, we have

$$\overline{(f * f)}(2n) = (f * f + f * \bar{f} + \bar{f} * f)(2n)$$

$$(2n-1)(f * f)(2n) = (f * f)(2n) + 2(f * \bar{f})(2n).$$

and the corollary follows.

3. PRELIMINARY IDENTITIES

Throughout the paper, we write for $s > 1$

$$\sigma(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}, \quad t(s) = \sum_{n=0}^{\infty} (2n+1)^{-s}, \quad \zeta(0) = -\tfrac{1}{2},$$

$\sigma(0) = \frac{1}{2}$ and $t(0) = 0$.

Clearly for $s = 0$ and any $s > 1$

$$\sigma(s) = (1 - 2^{1-s})\, \zeta(s) \qquad \ldots(3.1)$$

and

$$t(s) = \tfrac{1}{2}(\zeta(s) + \sigma(s)). \qquad \ldots(3.2)$$

Theorem 3.1—For $n \geq 1$, we have

$$6\,(\bar{\zeta} * \bar{\zeta})\,(2n) = (2n-1)\,(2n+1)\,(n-3)\,\zeta\,(2n) + 12\,\zeta\,(2)\,(2n-3)\,\zeta\,(2n-2).$$

PROOF : It is wellknown (cf. Knopp[5], section 24) that for $|z| < 1$, $z \neq 0$

$$\pi \cot \pi z = \frac{1}{z} - 2\sum_{n=1}^{\infty} \zeta\,(2n)\,z^{2n-1} \qquad \ldots(3.3)$$

so that by differentiation

$$\pi^2 z^2 \operatorname{cosec}^2 \pi z = 1 + 2\sum_{n=1}^{\infty} \bar{\zeta}\,(2n)\,z^{2n}. \qquad \ldots(3.4)$$

However, if $f^{(k)}(z)$ denotes the kth derivative of $f(z)$, then

$$6\pi^4\,(\operatorname{cosec}^4(\pi z))\,z^4 = -z^4\,(\pi \cot \pi z)^{(3)} - 4\pi^2 z^4\,(\pi \cot \pi z)^{(1)}.$$

Now on using (3.4) for the left-hand side and (3.3) for the right hand side and equating the corresponding coefficients in the resulting power series, we obtain the theorem.

Remark 3.1 : By Euler's formula (1.2), Theorem 3.1 is equivalent to

$$-\frac{B_{2k}}{k} = \frac{3\,(k-1)\,(2k-3)}{(2k+1)\,(k-3)} \sum_{l=2}^{k-2} \binom{2k-4}{2l-2} \frac{B_{2l}}{l}\,\frac{B_{2(k-l)}}{k-l}, \quad k \geq 4.$$

In this form, the result could be found in Rademacher[9] (p. 124, *l.* 3) whose proof is based on the nonlinear differential equation satisfied by the Weierstrass' p-function and the Fourier expansions of Eisenstein series of order $2k$. A trivial misprint occurs in Rademacher[9] (p. 124, l.3), in that the $-$ sign on the left is missing.

Theorem 3.2—For $n \geq 1$, we have

$$2\,(\bar{\zeta} * \sigma)\,(2n) = n\,(2n-3)\,\sigma\,(2n) - \bar{\zeta}\,(2n) + 3\zeta\,(2)\,\sigma\,(2n-2).$$

PROOF : Since $\operatorname{cosec}\theta = \cot\frac{\theta}{2} - \cot\theta$, we have by (3.3)

$$\pi z \operatorname{cosec}(\pi z) = 1 + 2\sum_{n=1}^{\infty} \sigma\,(2n)\,z^{2n}. \qquad \ldots(3.5)$$

However

$$2\pi^3 z^3 \operatorname{cosec}^3(\pi z) = z^3\,(\pi \operatorname{cosec}(\pi z))^{(2)} + \pi^3 z^3 \operatorname{cosec}(\pi z).$$

Now the left-hand side is $2(\pi^2 z^2 \operatorname{cosec}^2 \pi z)(\pi z \operatorname{cosec} \pi z)$ and hence on using (3.4) and (3.5) for the left-hand side and (3.5) for the right-hand side and equating the corresponding coefficients in the resulting power series, we obtain the theorem.

Theorem 3.3—For $n \geqslant 1$, we have

$$6(\bar{\zeta} * \bar{\sigma})(2n) = 3\bar{\zeta}(2n) + n(2n-5)\bar{\sigma}(2n) + 3\zeta(2)\bar{\sigma}(2n-2).$$

PROOF : By (3.5), we have on differentiation

$$\pi^2 z^2 \operatorname{cosec} \pi z \cot \pi z = 1 - 2 \sum_{n=1}^{\infty} \bar{\sigma}(2n) z^{2n}. \qquad \ldots(3.6)$$

However since

$$6\pi^3 \operatorname{cosec}^3 \pi z \cot \pi z = -(\operatorname{cosec} \pi z)^{(3)} - \pi(\pi \operatorname{cosec} \pi z)^{(1)}$$

and $\operatorname{cosec}^3 \pi z \cot \pi z = (\operatorname{cosec}^2 \pi z)(\operatorname{cosec} \pi z \cot \pi z)$, the theorem follows from (3.4), (3.5) and (3.6).

Theorem 3.4—For $n \geqslant 1$, we have

$$6(\bar{\sigma} * \bar{\sigma})(2n) = 6\bar{\sigma}(2n) + (n-1)(2n-3)\bar{\zeta}(2n) - 6\zeta(2)\bar{\zeta}(2n-2).$$

PROOF : By (3.6), we have

$$(1 - 2 \sum_{n=1}^{\infty} \bar{\sigma}(2n) z^{2n})^2 = \pi^4 z^4 \operatorname{cosec}^2 \pi z \cot^2 \pi z$$

$$= \frac{-z^4}{6}(\pi \cot \pi z)^{(3)} + \frac{\pi^2 z^4}{3}(\pi \cot \pi z)^{(1)}$$

so that the theorem follows upon using (3.3) on the right side of the above and equating the corresponding coefficients in the resulting power series.

Remark 3.2: We believe that Theorems 3.2 through 3.4 are all new.

4. MAIN RESULTS

In this section, we obtain closed evaluations for $(f * g)(2n)$, $n \geqslant 2$, where $f, g \in \{\zeta, \bar{\zeta}, \sigma, \bar{\sigma}, t, \bar{t}\}$ and $(f * g * h)(2n)$, $n \geqslant 3$ where $f, g, h \in \{\zeta, \sigma, t\}$, some of which have been discussed in section 3.

Theorem 4.1—For $n \geqslant 2$, we have

$$(\zeta * \zeta)(2n) = (n + \tfrac{1}{2})\zeta(2n) \qquad \ldots(4.1)$$

$$(\sigma * \sigma)(2n) = \tfrac{1}{2}\bar{\zeta}(2n) - \sigma(2n) \qquad \ldots(4.2)$$

$$(\sigma * \zeta)(2n) = n\,\sigma(2n) - \tfrac{1}{2}\zeta(2n) \qquad \ldots(4.3)$$

$$(t * \zeta)(2n) = nt(2n) \qquad \ldots(4.4)$$

$$(t * \sigma)(2n) = (n-1)\,t(2n) \qquad \ldots(4.5)$$

and

$$(t * t)(2n) = (n - \tfrac{1}{2})\,t(2n). \qquad \ldots(4.6)$$

PROOF : We note that (4.1) is simply (1.1) in the convolution language. Also (4.1) through (4.3) can be found in Nielsen[8] (pp. 37-51) and apparently go back to Euler. As Nielsen[8] remarks (p. 51), these respectively follow simply from

$$(\pi \cot \pi z)^{(1)} = -\pi^2 - \pi^2 \cot^2 \pi z$$

$$(\pi \cot \pi z)^{(1)} = -(\pi \operatorname{cosec} \pi z)^2$$

$$(\pi \operatorname{cosec} \pi z)^{(1)} = -(\pi \operatorname{cosec} \pi z)(\pi \cot \pi z)\,,$$

and the series representations (3.3) and (3.5). By a similar argument Ramanujan[10] (p. 2) provide (1.3) (which is equivanent to (1.1)) and various other results.

The identities (4.4) through (4.6) follow respectively from (4.1) through (4.3) in view of (3.2)

Theorem 4.2—For $n \geqslant 2$, we have

$$(\zeta * \bar{\zeta})(2n) = \frac{(n-1)(2n+1)}{2}\,\zeta(2n) \qquad \ldots(4.7)$$

$$(\sigma * \bar{\sigma})(2n) = (n-1)\,(\tfrac{1}{2}\,\bar{\zeta}(2n) - \sigma(2n)) \qquad \ldots(4.8)$$

$$(t * \bar{t})(2n) = \frac{(n-1)(2n-1)}{2}\,t(2n) \qquad \ldots(4.9)$$

$$2(\zeta * \bar{\sigma})(2n) = n\,\bar{\sigma}(2n) + \zeta(2n) - 3\zeta(2)\,\sigma(2n-2) \qquad \ldots(4.10)$$

$$4(\zeta * \bar{t})(2n) = n\bar{\sigma}(2n) + n\bar{\zeta}(2n) - 3\zeta(2)\,\sigma(2n-2) \qquad \ldots(4.11)$$

$$4(\bar{\zeta} * t)(2n) = 2n(2n-3)\,t(2n) + 3\zeta(2)\,\sigma(2n-2) \qquad \ldots(4.12)$$

$$4(\sigma * t)(2n) = 2(n-2)\,\bar{t}(2n) + 3\zeta(2)\,\sigma(2n-2) \qquad \ldots(4.13)$$

$$4(\bar{\sigma} * t)(2n) = 2(2n^2 - 3n + 2)\,t(2n) - 3\zeta(2)\,\sigma(2n-2) \qquad \ldots(4.14)$$

$$12(\bar{\zeta} * \bar{t})(2n) = 2n(2n-5)\,\bar{t}(2n) + 3\zeta(2)\{4\bar{\zeta}(2n-2) + \bar{\sigma}(2n-2)\} \qquad \ldots(4.15)$$

$$12\,(\bar{\sigma} * \bar{t})\,(2n) = n\,(2n-5)\,\bar{\sigma}\,(2n) + 3\,\bar{\zeta}\,(2n) + 3\zeta\,(2)\,\bar{\sigma}\,(2n-2) + 6\bar{\sigma}\,(2n) + (n-1)\,(2n-3)\bar{\zeta}\,(2n) - 6\zeta\,(2)\bar{\zeta}\,(2n-2) \quad \ldots(4.16)$$

and

$$6\,(\bar{t} * \bar{t})\,(2n) = (n-1)\,(2n-3)\,\bar{t}\,(2n) + 3\zeta\,(2)\,\bar{t}\,(2n-2). \quad \ldots(4.17)$$

PROOF : The identities (4.7), (4.8) and (4.9) follow respectively from (4.1), (4.2), and (4.3) and Corollary 2.1.

(4.10) follows from (4.3). Theorem 3.2 and Theorem 2.1

(4.11) follows from (4.7) and (4.10).

(4.12) follows from (4.4) and (4.11).

(4.13) follows from (3.2), (4.8) and Theorem 3.2.

(4.14) follows from Theorem 2.1, (4.5) and (4.13).

(4.15) follows from Theorems 3.1 and 3.3 and (3.2).

(4.16) follows from Theorems 3.3 and 3.4 and (3.2).

Finally (4.17) follows from (4.15) and (4.16).

Theorem 4.3—For $n \geqslant 3$, we have

$$4\,(\zeta * \zeta * \zeta)\,(2n) = (n+1)\,(2n+1)\,\zeta\,(2n) - 6\zeta\,(2)\,\zeta\,(2n-2) \quad \ldots(4.18)$$

$$4\,(\zeta * \zeta * \sigma)\,(2n) = (2n+1)\,\{n\sigma\,(2n) - \zeta\,(2n)\} - 3\zeta\,(2)\,\sigma\,(2n-2) \quad \ldots(4.19)$$

$$4\,(\zeta * \zeta * t)\,(2n) = n\,(2n+1)\,t\,(2n) - \tfrac{3}{2}\,\zeta\,(2)\,\zeta\,(2n-2) - 3\zeta\,(2)\,t\,(2n-2) \quad \ldots(4.20)$$

$$4\,(\zeta * \sigma * \sigma)\,(2n) = (2n^2 - n + 1)\,\zeta\,(2n) - 4n\sigma\,(2n) \quad \ldots(4.21)$$

$$4\,(\zeta * \sigma * t)\,(2n) = n\,(2n-3)\,t\,(2n) - \tfrac{3}{2}\,\zeta\,(2)\,\sigma\,(2n-2) \quad \ldots(4.22)$$

$$4\,(\zeta * t * t)\,(2n) = n\bar{t}\,(2n) - 3\,\zeta\,(2)\,t\,(2n-2) \quad \ldots(4.23)$$

$$4\,(\sigma * \sigma * \sigma)\,(2n) = (2n^2 - 3n + 4)\,\sigma\,(2n) - 3\bar{\zeta}\,(2n) + 3\zeta\,(2)\,\sigma\,(2n-2) \quad \ldots(4.24)$$

$$4\,(\sigma * \sigma * t)\,(2n) = (2n^2 - 7n + 4)\,t\,(2n) + \tfrac{3}{2}\zeta\,(2)\,\sigma\,(2n-2) \quad \ldots(4.25)$$

$$4\,(\sigma * t * t)\,(2n) = (n-2)\,\bar{t}\,(2n) \qquad \ldots(4.26)$$

and

$$4\,(t * t * t)\,(2n) = (n-1)\,\bar{t}\,(2n) - \tfrac{3}{2}\zeta\,(2)\,t\,(2n-2). \qquad \ldots(4.27)$$

PROOF : We prove (4.18) in detail. The proofs of (4.19) through (4.27) being similar, we only sketch them.

To prove (4.18), we have by (4.1) and (4.7)

$$(\zeta * \zeta * \zeta)\,(2n) = \sum_{\substack{a+b=n \\ a\geqslant 1,\ b\geqslant 1}} \zeta\,(2a)\,(\zeta * \zeta)\,(2b)$$

$$= \sum_{\substack{a+b=n \\ a\geqslant 1, b\geqslant 2}} \zeta\,(2a)\left(b + \frac{1}{2}\right)\zeta\,(2b) = \sum_{\substack{a+b=n \\ a\geqslant 1, b\geqslant 1}} \left\{\zeta\,(2a) \times \left(\frac{2b-1}{2} + 1\right)\zeta\,(2b)\right\} - \frac{3}{2}\,\zeta(2)\,\zeta\,(2n-2)$$

$$= \tfrac{1}{2}\,(\zeta * \bar{\zeta})\,(2n) + (\zeta * \zeta)\,(2n) - \tfrac{3}{2}\,\zeta\,(2)\,\zeta\,(2n-2)$$

$$= \frac{1}{2}\,\frac{(n-1)\,(2n+1)}{2}\,\zeta\,(2n) + (n + \tfrac{1}{2})\,\zeta\,(2n) - \tfrac{3}{2}\,\zeta\,(2)\,\zeta\,(2n-2)$$

$$= \frac{(n+1)\,(2n+1)}{4}\,\zeta\,(2n) - \tfrac{3}{2}\,\zeta\,(2)\,\zeta\,(2n-2).$$

This proves (4.18).

(4.19) follows from (4.3), (4.10) and (4.1).

(4.20) follows from (4.18), (4.19) and (3.2).

(4.21) follows from (4.2), (4.7) and (4.3).

(4.22) follows from (4.19) and (4.21).

(4.23) follows from (4.20) and (4.22).

(4.24) follows from (4.2) and Theorem 3.2.

(4.25) follows from (4.21) and (4.24).

(4.26) follows from (4.22) and (4.25).

(4.27) follows from (4.23) and (4.26).

This completes the proof of Theorem 4.3.

5. Estermann's and Williams' Identities

In this section, we give simple proofs for certain identities due to Estermann[4] and Williams[17] and prove some more identities.

Theorem 5.1—For $n \geq 1$, we have

$$2t(2n+2) = 3\zeta(2)\,\sigma(2n) - 4\sum_{i=1}^{n}(2i-1)\,2^{-2i}\,\zeta(2i)\,t(2n+2-2i).$$

Proof : We give two proofs; the first is in the spirit of Section 3 and the second uses some results of Section 4.

By (3.3) and (3.5), we have

$$\pi z \operatorname{cosec} \pi z - \pi z \cot \pi z = 4\sum_{n=1}^{\infty} t(2n)\,z^{2n}.$$

However by (3.4) we have

$$\frac{\pi^2 z^2}{4}\operatorname{cosec}^2\left(\frac{\pi z}{2}\right) = 1 + 2\sum_{n=1}^{\infty}(2n-1)\,2^{-2n}\,\zeta(2n)\,z^{2n}.$$

Now substituting the power series representations from above and (3.5) in the identity

$$(\pi z \operatorname{cosec} \pi z - \pi z \cot \pi z)\left(\frac{\pi^2 z^2}{4}\operatorname{cosec}^2\left(\frac{\pi z}{2}\right)\right) = \frac{\pi^2}{2}(\pi z \operatorname{cosec} \pi z)\,z^2$$

and equating the coefficients of z^{2n+1} on both sides, we complete the proof.

For a second proof, we note that the right-hand side of the assertion is

$$3\zeta(2)\,\sigma(2n) + 4\sum_{i=1}^{n}(2i-1)(1-2^{-2i})\,\zeta(2i)\,t(2n+2-2i) - 4\sum_{i=1}^{n}(2i-1)\,\zeta(2i)\,t(2n+2-2i)$$

$$= 3\zeta(2)\,\sigma(2n) + 4\,(\bar{t} * t)(2n+2) - (\bar{\zeta} * t)(2n+2)$$

and the theorem follows from (4.9) and (4.12).

Remark 5.1 : Theorem 5.1 is originally due to Estermann[4] whose proof is based on the rearrangement of absolutely convergent series.

Theorem 5.2—For $s > 0$, let $\eta(s) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}$. Then for $n \geq 1$

$$\eta(1)\,\eta(2n-1)+\eta(3)\,\eta(2n-3)+\ldots+\eta(2n-1)\,\eta(1)$$
$$=(n-\tfrac{1}{2})\,t(2n). \qquad \ldots(5.1)$$
$$\eta(1)\,t(2n)+\eta(3)\,t(2n-2)+\ldots+\eta(2n-1)\,t(2)$$
$$=n\,\eta(2n+1) \qquad \ldots(5.2)$$

PROOF : Since $\tan\theta=\cot\theta-2\cot 2\theta$, we have by (3.3)

$$\pi\tan\pi z=\sum_{n=1}^{\infty}2(2^{2n}-1)\,\zeta(2n)\,z^{2n-1},\ |z|<\tfrac{1}{2}.$$

Also we recall the well known[5] (Section 24) result

$$\pi\sec\pi z=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)}{(n-\frac{1}{2})^2-z^2},\ |z|<\tfrac{1}{2}$$

so that we easily obtain

$$\sum_{n=1}^{\infty}\eta(2n-1)\,z^{2n-1}=\frac{\pi z}{4}\sec\left(\frac{\pi z}{2}\right).$$

Now (5.1) and (5.2) respectively follow on using the above power series representations in the identities

$$\left(\frac{\pi}{4}\sec\frac{\pi z}{2}\right)^2=\frac{\pi}{8}\left(\tan\frac{\pi z}{2}\right)^{(1)}$$
$$(\sec\pi z)(\pi\tan\pi z)=(\sec\pi z)^{(1)}$$

and equating the corresponding coefficients.

Remark 5.2 : The identity (5.1) is originally due to Williams[17] (Theorem 2) and (5.2) is believed to be new.

Theorem 5.3—For $n\geqslant 4$, we have

$$24(\zeta*\zeta*\zeta*\zeta)2n=(n+1)(2n+1)(2n+3)\,\zeta(2n)$$
$$-48n\zeta(2)\,\zeta(2n-2).$$

PROOF : By (4.1), we have

$$24(\zeta*\zeta*\zeta*\zeta)(2n)=24\sum_{a+b=n}(\zeta*\zeta)(2a)(\zeta*\zeta)(2b)$$
$$=24\sum_{\substack{a+b=n\\ a\geqslant 2,b\geqslant 2}}(a+\tfrac{1}{2})\zeta(2a)(b+\tfrac{1}{2})\,\zeta(2b)$$

(equation continued on p. 1185)

$$= 6 \sum_{\substack{a+b=n \\ a\geqslant 1,\ b\geqslant 1}} \{(2a-1)+2\}\zeta(2a)\{(2b-1)+2\}\zeta(2b)$$

$$- 36(2n-1)\zeta(2)\zeta(2n-2)$$

$$= 6 \sum_{a+b=n} (\overline{\zeta}(2a) + 2\zeta(2a))(\overline{\zeta}(2b) + 2\zeta(2b))$$

$$- 36(2n-1)\zeta(2)\zeta(2n-2)$$

$$= 6(\overline{\zeta} * \overline{\zeta})(2n) + 24(\zeta * \overline{\zeta})(2n) + 24(\zeta * \zeta)(2n)$$

$$- 36(2n-1)\zeta(2)\zeta(2n-2).$$

Now the theorem follows from Theorem 3.1, (4.7) and (4.1).

Remark 5.3 : For $f \in F$ and $k \geqslant 2$ let f^{*k} denote the convolution product of k copies of f. While an evaluation of ζ^{*2} [given in (4.1)] is a fairly classical result, it may be noted that (4.18) and Theorem 5.3 provide closed evaluations of ζ^{*3} and ζ^{*4}. It may be of interest to obtain a closed evaluation of $\zeta^{*k}(2n)$ for $n \geqslant k \geqslant 5$.

Remark 5.4 : Using a series inversion formula, Titchmarsh[14] [(8.51) and (8.52)] proved the following identities

$$(2^{4k}-1)\zeta(4k)$$

$$= \frac{4}{\pi^2} \sum_{p=1}^{k} \sum_{q=1}^{k} (2^{2p}-1)(2^{2q}-1)\zeta(2p)\zeta(2q)\zeta(4k-2p-2q+2)$$

and

$$(2^{4k+2}-1)\zeta(4k+2) = \frac{2}{\pi^2}(2^{2k+2}-1)^2\zeta^2(2k+2)$$

$$+ \frac{4}{\pi^2} \sum_{p=1}^{k} \sum_{q=1}^{k} (2^{2p}-1) = (2^{2q}-1)\zeta(2p)\zeta(2q)$$

$$\times \zeta(4k-2p-2q+4).$$

The authors were not successful in giving proofs of these identities by the method of this paper.

ACKNOWLEDGEMENT

The authors thank Professor P. J. deDoelder, Eindhoven, Netherlands, for checking through an earlier version and pointing out some inaccuracies.

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