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PSEUDO *H*-PROJECTIVE *n*-RECURRENT MANIFOLD WITH *KH*-STRUCTURE

R. S. MISHRA, F. N. A.*, S. B. PANDEY+ AND U. SHARMA++

Department of Mathematics, Banaras Hindu University, Varanasi

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In this paper, we have defined pseudo *H*-projective *n*-recurrent manifold equipped with *KH*-structure and studied its properties with respect to other curvature tensors.

1. INTRODUCTION

We consider a $2m$-dimensional, C^∞, manifold M_{2m}. Let there exist in M_{2m}, a vector-valued, C^∞, linear function F, such that

$$\bar{\bar{X}} = a^2 X \qquad \ldots(1.1a)$$

for arbitrary vector field X and complex constant a, where

$$\bar{X} \underline{\underline{\text{ def }}} F(X). \qquad \ldots(1.1b)$$

Then F is said to give to M_{2m}, a differentiable structure briefly GF-structure defined by algebraic equations (1.1) and the manifold is called GF-structure manifold[1].

Let the manifold M_{2m} be endowed with the Hermitian metric g such that

$$g(\bar{X}, \bar{Y}) + a^2 g(X, Y) = 0. \qquad \ldots(1.2)$$

Then we say that (F, g) gives to M_{2m} a Hermite structure or H-structure.

Agreement 1.1—In what follows and above, the equations containing $X, Y, Z, \ldots$, etc. hold for arbitrary vectors $X, Y, Z, \ldots$, etc. in M_{2m}.

We consider a (0, 2) type, tensor $'F$ in M_{2m} equipped with the H-structure, such that

$$'F(X, Y) \underline{\underline{\text{ def }}} g(\bar{X}, Y) = -g(X, \bar{Y}). \qquad \ldots(1.3)$$

Then the following equations hold :

$$'F(X, Y) + 'F(Y, X) = 0 \qquad \ldots(1.4a)$$

* V. C. Lucknow University Lucknow at present.

+ Department of Mathematics, Kumaun University, Almora (U.P.).

++ G. G. I. C. Almora

$$'F(\bar{X}, Y) + 'F(X, \bar{Y}) = 0 \qquad \ldots(1.4b)$$

$$'F(\bar{X}, \bar{Y}) + a^2\, 'F(X, Y) = 0. \qquad \ldots(1.4c)$$

In M_{2m} equipped with H-structure, if D be the Riemannian Connection, such that

$$(D_X F)(Y) = (VF)(Y, X) = 0. \qquad \ldots(1.5)$$

V being the operator of covariant differentiation, is satisfied then M_{2m} is said to be with KH-structure.

In the following M_{2m} is supposed to be equipped with KH-structure, unless stated otherwise.

Let K and Ric denote the curvature tensor and Ricci tensor in M_{2m}. Then the following equations are true[6] :

$$K(X, Y, \bar{Z}) = \overline{K(X, Y, Z)} \qquad \ldots(1.6a)$$

$$\overline{K(X, Y, \bar{Z})} = a^2\, K(X, Y, Z) \qquad \ldots(1.6b)$$

$$'K(X, Y, \bar{Z}, \bar{T}) = -a^2\, 'K(X, Y, Z, T) = 'K(\bar{X}, \bar{Y}, Z, T) \qquad \ldots(1.6c)$$

where

$$'K(X, Y, Z, T) \underline{\text{ def }} g(K(X, Y, Z), T) \qquad \ldots(1.6d)$$

$$\text{Ric}(Y, Z) = g(r(Y), Z),\ \text{Ric}(Y, Z) = (C_1^1\, K)(Y, Z) \qquad \ldots(1.6e)$$

$$\text{Ric}(\bar{X}, \bar{Y}) = -a^2\, \text{Ric}(X, Y),\ (C_1^1\, r) = R \qquad \ldots(1.6f)$$

$$\text{Ric}(\bar{X}, Y) = -\text{Ric}(X, \bar{Y}). \qquad \ldots(1.6g)$$

r is the self-adjoint Ricci map C_1^1 is the contraction w.r.t. first slot.

The Pseudo H-projective curvature tensor in M_{2m}, where $a \neq 0$, is given by[4] :

$$P(X, Y, Z) = K(X, Y, Z) - \frac{1}{2a^2(m+1)}[a^2\, \text{Ric}(Y, Z)X - a^2\, \text{Ric}(X, Z)Y - \text{Ric}(\bar{Y}, Z)\bar{X} + \text{Ric}(\bar{X}, Z)\bar{Y} - 2\, \text{Ric}(X, \bar{Y})\bar{Z}] \qquad \ldots(1.7a)$$

and

$$'P(X, Y, Z, T) \underline{\text{ def }} g(P(X, Y, Z), T). \qquad \ldots(1.7b)$$

Let the projective curvature tensor W, conformal curvature tensor C, conharmonic curvature tensor L, concircular curvature tensor V, H-conharmonic curvature tensor S and Bochner curvature tensor B be respectively given by

$$W(X, Y, Z) = K(X, Y, Z) - \frac{1}{(2m-1)}[\text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y]. \quad \ldots(1.8)$$

$$C(X, Y, Z) = K(X, Y, Z) - \frac{1}{2(m-1)}[\text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y - g(X, Z)r(Y) + g(Y, Z)r(X)] + \frac{R}{2(2m-1)(m-1)}[g(Y, Z)X - g(X, Z)Y]. \quad \ldots(1.9)$$

$$L(X, Y, Z) = K(X, Y, Z) - \frac{1}{2(m-1)}[\text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y + g(Y, Z)r(X) - g(X, Z)r(Y)]. \quad \ldots(1.10)$$

$$V(X, Y, Z) = K(X, Y, Z) - \frac{R}{2m(2m-1)}[g(Y, Z)X - g(X, Z)Y]. \quad \ldots(1.11)$$

$$S(X, Y, Z) = K(X, Y, Z) + \frac{1}{2(m+2)}[\text{Ric}(X, Z)Y - \text{Ric}(Y, Z)X + \text{Ric}(\bar{X}, Z)\bar{Y} - \text{Ric}(\bar{Y}, Z)\bar{X} + 2\,\text{Ric}(\bar{X}, Y)\bar{Z} + g(X, Z)r(Y) - g(Y, Z)r(X) + g(\bar{X}, Z)r(\bar{Y}) - g(\bar{Y}, Z)r(\bar{X}) + 2g(\bar{X}, Y)r(\bar{Z})]. \quad \ldots(1.12)$$

$$B(X, Y, Z) = K(X, Y, Z) + \frac{1}{2(m+2)}[\text{Ric}(X, Z)Y - \text{Ric}(Y, Z)X + g(X, Z)r(Y) - g(Y, Z)r(X) + \text{Ric}(\bar{X}, Z)\bar{Y} - \text{Ric}(\bar{Y}, Z)\bar{X} + g(\bar{X}, Z)r(\bar{Y}) - g(\bar{Y}, Z)r(\bar{X}) + 2\,\text{Ric}(\bar{X}, Y)\bar{Z} + 2g(\bar{X}, Y)r(\bar{Z})] - \frac{R}{4(m+1)(m+2)}[g(X, Z)Y - g(Y, Z)X + g(\bar{X}, Z)\bar{Y}, -g(\bar{Y}, Z)\bar{X} + 2g(\bar{X}, Y)(\bar{Z})]. \quad \ldots(1.13)$$

Eliminating K from the equations (1.8), (1.9), (1.10), (1.11), (1.12) and (1.13) with the help of (1.7a), we have

$$P(X, Y, Z) = W(X, Y, Z) + \frac{3}{2(2m-1)(m+1)}[\text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y] - \frac{1}{2a^2(m+1)}$$

(*equation continued on p.* 1190)

$$[\text{Ric}\,(Y, \bar{Z})\,\bar{X} - \text{Ric}\,(X, \bar{Z})\,\bar{Y} - 2\,\text{Ric}\,(X, \bar{Y})\,\bar{Z}]. \qquad \ldots(1.14)$$

$$P(X, Y, Z) = C(X, Y, Z) + \frac{1}{(m^2 - 1)}[\text{Ric}\,(Y, Z)\,X) - \text{Ric}\,(X, Z)\,Y]$$
$$+ \frac{1}{2(m+1)}[g(Y, Z)\,r(X) - g(X, Z)\,r(Y)]$$
$$- \frac{R}{2(m-1)(2m-1)}[g(Y, Z)\,X - g(X, Z)\,Y]$$
$$- \frac{1}{2(m+1)a^2}[\text{Ric}\,(Y, \bar{Z})\,\bar{X} - \text{Ric}\,(X, \bar{Z})\,\bar{Y}$$
$$- 2\,\text{Ric}\,(X, \bar{Y})\,\bar{Z}]. \qquad \ldots(1.15)$$

$$P(X, Y, Z) = L(X, Y, Z) + \frac{1}{(m^2 - 1)}[\text{Ric}\,(Y, Z)\,X - \text{Ric}\,(X, Z)\,Y]$$
$$+ \frac{1}{2(m+1)}[g(Y, Z)\,r(X) - g(X, Z)\,r(Y)]$$
$$- \frac{1}{2(m+1)a^2}[\text{Ric}\,(\bar{Y}, Z)\,\bar{X} - \text{Ric}\,(\bar{X}, Z)\,\bar{Y}$$
$$- 2\,\text{Ric}\,(X, \bar{Y})\,\bar{Z}] \qquad \ldots(1.16)$$

$$P(X, Y, Z) = V(X, Y, Z) + \frac{R}{2m(2m-1)}[g(Y, Z)\,X - g(X, Z)\,Y]$$
$$- \frac{1}{2(m+1)a^2}[a^2\,\text{Ric}\,(Y, Z)\,X - a^2\,\text{Ric}\,(X, Z)\,Y$$
$$+ \text{Ric}\,(Y, \bar{Z})\,\bar{X} - \text{Ric}\,(X, \bar{Z})\,\bar{Y} - 2\,\text{Ric}\,(X, \bar{Y})\,\bar{Z}]. \qquad \ldots(1.17)$$

$$P(X, Y, Z) = S(X, Y, Z) - \frac{1}{2(m+1)(m+2)}[\text{Ric}\,(Y, Z)\,X$$
$$- \text{Ric}\,(X, Z)\,Y]$$
$$- \frac{(m+2) + a^2(m+1)}{2a^2(m+1)(m+2)}[\text{Ric}\,(\bar{X}, Z)\,\bar{Y} - \text{Ric}\,(\bar{Y}, Z)\,\bar{X}$$
$$+ 2\,\text{Ric}\,(\bar{X}, Y)\,\bar{Z}]$$
$$- \frac{1}{2(m+2)}[g(X, Z)\,r(Y) - g(Y, Z)\,r(X)$$
$$+ g(\bar{X}, Z)\,r(\bar{Y}) - g(\bar{Y}, Z)\,r(\bar{X}) + 2g(\bar{X}, Y)\,r(\bar{Z})]. \qquad \ldots(1.18)$$

$$P(X, Y, Z) = B(X, Y, Z) - \frac{1}{2(m+1)(m+2)} [\text{Ric}(Y, Z) X - \text{Ric}(X, Z) Y] + \frac{(m+2) + a^2(m+1)}{2a^2(m+1)(m+2)}$$

$$[\text{Ric}(\bar{Y}, Z) \bar{X} - \text{Ric}(\bar{X}, Z) \bar{Y} - 2 \text{Ric}(\bar{X}, Y) \bar{Z}]$$

$$- \frac{1}{2(m+2)} [g(X, Z) r(Y) - g(Y, Z) r(X)$$

$$+ g(\bar{X}, Z) r(\bar{Y}) - g(\bar{Y}, Z) r(\bar{X}) + 2g(\bar{X}, Y) r(\bar{Z})]$$

$$+ \frac{R}{4(m+1)(m+2)} [g(X, Z) Y - g(Y, Z) X$$

$$+ g(\bar{X}, Z) \bar{Y} - g(\bar{Y}, Z) \bar{X} + 2g(\bar{X}, Y) \bar{Z}]. \quad \ldots(1.19)$$

The manifold M_{2m} is said to be recurrent[2], if

$$(\nabla K)(X, Y, Z, U_1) = A_1(U_1) K(X, Y, Z). \quad \ldots(1.20a)$$

It is said to be Ricci recurrent, if

$$(\nabla \text{Ric})(Y, Z, U_1) = A_1(U_1) \text{Ric}(Y, Z) \quad \ldots(1.20b)$$

which gives

$$(\nabla r)(Z, U_1) = A_1(U_1) r(Z) \quad \ldots(1.20c)$$

$$(\nabla R)(U_1) = A_1(U_1) R \quad \ldots(1.20d)$$

where A_1 is a non-vanishing, C^∞, 1-form.

The manifold M_{2m} is said to be birecurrent[5], if

$$(\nabla\nabla K)(X, Y, Z, U_1, U_2) = A_2(U_1, U_2) K(X, Y, Z). \quad \ldots(1.21a)$$

It is said to be Ricci birecurrent, if

$$(\nabla\nabla \text{Ric})(Y, Z, U_1, U_2) = A_2(U_1, U_2) \text{Ric}(Y, Z) \quad \ldots(1.21b)$$

where A_2 is a non-vanishing, C^∞, 2-form such that

$$A_2(U_1, U_2) = (\nabla A_1)(U_1, U_2) + A_1(U_1) A_2(U_2). \quad \ldots(1.21c)$$

The manifold M_{2m} is said to be n-recurrent[3], if

$$(\nabla \ldots \nabla\nabla K)(X, Y, Z, U_1, U_2, \ldots, U_n) = A_n(U_1, U_2, \ldots U_n) K(X, Y, Z). \quad \ldots(1.22a)$$

It is said to be Ricci n-recurrent, if

$$(\nabla \ldots \nabla\nabla \text{ Ric})\,(Y, Z, U_1, U_2, \ldots, U_n) = A_n\,(U_1, U_2, \quad, U_n)\,\text{Ric}\,(Y, Z) \qquad \ldots(1.22b)$$

which gives

$$(\nabla \ldots \nabla\nabla r)\,(Z, U_1, U_2, \ldots U_n) = A_n\,(U_1, U_2, \ldots, U_n)\,r\,(Z). \qquad \ldots(1.22c)$$

$$(\nabla \ldots \nabla\nabla R)\,(U_1, U_2, \ldots, U_n) = A_n\,(U_1, U_2, \ldots, U_n)\,R \qquad \ldots(1.22d)$$

where A_n is non-vanishing C^∞, n-form, such that

$$A_n\,(U_1, U_2, \ldots, U_n) = (\nabla\, A_{n-1}\, \chi\, U_1, U_2, \ldots, U_n) + A_{n-1}\,(U_1, U_2, \ldots, U_{n-1})\,A_1 U_n. \qquad \ldots(1.22e)$$

Equations (1.22) show that if the manifold M_{2m} is n-recurrent then it is Ricci n-reccurrent.

Let Q, a vector valued, trilinear function be any one of the curvature tensors W, C, L, V, S or B then M_{2m} is said to be Q n-reccurrent, if

$$(\nabla \ldots \nabla\nabla\, Q)\,(X, Y, Z, U_1, U_2, \ldots, U_n) = A_n\,(U_1, U_2, \ldots, U_n)\,Q\,(X, Y, Z). \qquad \ldots(1.23)$$

The manifold M_{2m} is said to be n-symmetric, if

$$(\nabla \ldots \nabla\nabla\, K)\,(X, Y, Z, U_1, U_2, \ldots, U_n) = 0 \qquad \ldots(1.24)$$

which gives

$$(\nabla \ldots \nabla\nabla \text{ Ric})\,(Y, Z, U_1, U_2, \ldots, U_n) = 0. \qquad \ldots(1.25a)$$

or

$$(\nabla \ldots \nabla\nabla\, r)\,(Z, U_1, U_2, \ldots, U_n) = 0. \qquad \ldots(1.25b)$$

or

$$(\nabla \ldots \nabla\nabla\, R)\,(U_1, U_2, \ldots, U_n) = 0. \qquad \ldots(1.25c)$$

Equations (1.25) define that the manifold is Ricci n-symmetric.

The manifold M_{2m} is said to be Q n-symmetric, if

$$(\nabla \ldots \nabla\nabla\, Q)\,(X, Y, Z, U_1, U_2, \ldots, U_n) = 0.$$

2. Pseudo H-Projective Curvature Tensor

The manifold M_{2m} is said to be Pseudo H-projective n-recurrent, if

$$(\nabla \ldots \nabla\nabla\, P)\,(X, Y, Z, U_1, U_2, \ldots, U_n) = A_n\,(U_1, U_2, \ldots, U_n)\,P\,(X, Y, Z). \qquad \ldots(2.1)$$

The manifold M_{2m} is said to be Pseudo H-projective n-symmetric if

$$(\nabla \ldots \nabla\nabla\, P)\,(X, Y, Z, U_1, U_2, \ldots, U_n) = 0. \qquad \ldots(2.2)$$

Theorem 2.1—If the manifold M_{2m} be n-recurrent then it is Pseudo H-projective n-recurrent for the same proportionality form A_n.

PROOF : Differentiating covariantly eqn. (1.7a) with respect to $U_1, U_2, U_3,...,U_n$ successively, we get

$$\begin{aligned}(\nabla...\nabla\nabla P)(X, Y, Z, U_1, U_2,...,U_n) &= (\nabla...\nabla\nabla K)(X, Y, Z, U_1, U_2,...,U_n)\\ &- \frac{1}{2a^2(m+1)}[a^2(\nabla...\nabla\nabla \text{ Ric})(Y, Z, U_1 U_2,...,U_n) X\\ &- a^2(\nabla...\nabla\nabla \text{ Ric})(X, Z, U_1, U_2,...,U_n) Y\\ &- (\nabla...\nabla\nabla \text{ Ric})(\bar{Y}, Z, U_1, U_2, ,U_n) \bar{X}\\ &+ (\nabla \nabla\nabla \text{ Ric})(\bar{X}, Z, U_1, U_2, ,U_n) \bar{Y}\\ &- 2(\nabla...\nabla\nabla \text{ Ric})(X, \bar{Y}, U_1, U_2...,U_n) \bar{Z}]. \qquad ...(2.3)\end{aligned}$$

Since the manifold is n-recurrent, using the equations (1.22) and (1.7a) in (2.3), we get (2.1).

Hence the statement.

Theorem 2.2—If the manifold M_{2m} be Pseudo-H-projective n-recurrent and Ricci n-recurrent then it is n-recurrent for the same proportionality form A_n.

PROOF : Multiplying eqn. (1.7a) by $A_n(U_1, U_2,...,U_n)$ throughout, substracting the resulting equation from (2.3) and then using (2.1) and (1.22b), we get (1.22a).

Hence the statement.

Theorem 2.3—In the manifold M_{2m}, if any two of the following conditions hold then the third also holds for the same proportionality form A_n.

(i) It is Pseudo H-projective n-recurrent;

(ii) It is projective n-recurrent;

(iii) It is Ricci n-recurrent.

PROOF : Differentiating covariantly eqn. (1.14) with respect to $U_1, U_2,...,U_n$ successively, we get

$$\begin{aligned}(\nabla \nabla\Delta P)(X, Y, Z, U_1, U_2,...,U_n) &= (\nabla \nabla\nabla W)(X, Y, Z, U_1, U_2,...,U_n)\\ &+ \frac{3}{2(2m-1)(m+1)}[(\nabla... \nabla\nabla\text{Ric})(Y, Z, U_1, U_2,...,U_n) X\\ &- (\nabla \nabla\nabla\text{Ric})(X, Z, U_1, U_2,...,U_n) Y]\end{aligned}$$

(equation continued on p.1194)

$$- \frac{1}{2a^2(m+1)} [(\nabla \ldots \nabla\nabla \text{Ric}(Y, \bar{Z}\, U_1, U_2, \ldots, U_n)\, \bar{X}$$

$$- (\nabla \ldots \nabla\nabla \text{Ric})(X, \bar{Z}, U_1, U_2, \ldots, U_n)\, \bar{Y}$$

$$- 2(\nabla \ldots \nabla\nabla \text{Ric})(X, \bar{Y}, U_1, U_2, \ldots, U_n)\, \bar{Z}]. \qquad \ldots(2.4)$$

Multiplying eqn. (1.14) by $A_n(U_1, U_2, \ldots, U_n)$ throughout, substracting the resulting equation from (2.4) and then using the fact that the manifold is Pseudo H-projective n-recurrent and projective n-recurrent, we have

$$(\nabla \ldots \nabla\nabla \text{Ric})(Y, Z, U_1, U_2, \ldots, U_n) = A_n(U_1, U_2, \ldots, U_n)\, \text{Ric}(Y, Z)$$

which shows that the manifold is Ricci n-recurrent.

Similarly, it can be shown that if the manifold is either Pseudo H-projective n-recurrent and Ricci n-recurrent or projective n-recurrent and Ricci n-recurrent, then it is either projective n-recurrent or Pseudo H-projective n-recurrent.

Hence the statement.

Theorem 2.4—In the manifold M_m, if any two of the following conditions hold then the third also holds for the same proportionality from A_n.

(i) It is Pseudo H-projective n-recurrent;

(ii) It is conformal or conharmonic or concircular or H-conharmonic or Bochner n-recurrent;

(iii) It is Ricci n-recurrent;

The proof follows the pattern of the proof of Theorem 2.3.

Theorem 2.5—If the manifold M_{2m} be n-symmetric then it is Pseudo h-projective n-symmetric.

PROOF : Since the n-symmetric manifold is Ricci n-symmetric, using eqn. (1.24) and (1.25) in (2.3), we get (2.2).

Hence the statement.

Theorem 2.6—If the manifold M_{2m} be Pseudo H-projective n-symmetric and Ricci n-symmetric then it is n-symmetric.

PROOF : Using eqns. (2.2) and (1.25) in (2.3), we get (1.24).

Hence the statement.

Theorem 2.7—In the manifold M_{2m}, if any two of the following conditions hold then the third also holds :

(i) It is Pseudo H-projective n-symmetric;

(ii) It is projective n-symmetric;

(iii) It is Ricci n-symmetric.

PROOF : Replacing Q by W in (1.26) and then using the resulting equation and (2.2) in (2.4), we get

$$(\nabla \ldots \nabla\nabla \mathrm{Ric})(Y, Z, U_1, U_2, \ldots, U_n) = 0$$

which is (1.25a). The remaining part follows similarly.

Note 2.1 : Theorems of the type 2.7 can also be proved taking conformal or conharmonic or concircular or H-conharmonic or Bochner curvature tensors instead of projective curvature tensor in M_{2m}.

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