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ON A CLASS OF ORBIT CLOSURES WITH UNIQUE MINIMAL SETS

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A class of $0 - 1$ sequences whose orbit closures contain a unique minimal set is constructed and their unique ergodicity investigated. Besides finding necessary and sufficient conditions for unique ergodicity it is proved that when all points are quasi-regular the orbit closure is uniquely ergodic.

INTRODUCTION

A familiar class of almost periodic 0-1 sequences is constructed as follows[2]: Let $S = \{0,1\}$ and $S^* = \bigcup_{k \geqslant 1} S^k$; that is, S^* the set of all blocks in S. A map $\theta : \mathbb{N} \to S^*$ induces a map $\bar{\theta} : \mathbb{N}^{\mathbb{Z}} \to S^{\mathbb{Z}}$ once an origin is chosen. If the image of θ has only finitely many distinct blocks, then for any almost periodic point $\omega \in \mathbb{N}^{\mathbb{Z}}$, $\bar{\theta}(\omega)$ is an almost periodic 0-1 sequence. Further, the techniques of Jacobs and Keane[3] yield a sufficient condition for unique ergodicity of the shift on the orbit closure of $\bar{\theta}(\omega)$. In this note we study the 0-1 sequences arising from the above construction when the image of θ has infinitely many blocks. Here we can assume that θ is injective. If $\omega \in F^{\mathbb{Z}}$ for some finite subset F of $\mathbb{N}$ (that is, if ω involves only finitely many n) then the study of $\bar{\theta}(\omega)$ can be reduced to the above case. When θ is close to an almost periodic 0-1 sequence x_0 in the sense that for all but finitely many n, the block $\theta(n)$ occurs in x_0, for any almost periodic point $\omega \in \mathbb{N}^{\mathbb{Z}}$ such that $\omega \notin F^{\mathbb{Z}}$ for any finite subset of $\mathbb{N}$, $OC(\bar{\theta}(\omega))$ contains $OC(x_0)$ as the unique minimal subset. In this case, we will obtain explicit necessary and sufficient conditions on θ and ω so that $OC(\theta(\omega))$ has a unique shift invariant measure. We will also show that the shift is uniquely ergodic on $OC(\bar{\theta}(\omega))$ if and only if all points of $OC(\bar{\theta}(\omega))$ are quasi-regular.

In the literature some attention is paid to the class of dynamical systems with a unique minimal set. Dowker and Lederer[1] proved that if all points are also quasi-regular, then the system is either uniquely ergodic or admits infinitely many ergodic invariant measures. Katznelson and Weiss strengthen the above result by showing that when the system is not uniquely ergodic, it must admit uncountably many ergodic invariant measures. They also construct an example to show that the second possibility is not vacuous. However, it seems reasonable to conjecture that in 0-dimensional systems only the first possibility occurs. This turns out to be the case for the class of systems considered in this note. We remark however, that by our methods it does not seem possible to obtain stronger results towards the proof of the above conjecture.

1. Notations and Terminology

A subset $\Lambda \subset \mathbb{Z}$ is said to be relatively dense if there exists an integer $L > 0$ such that for any $z \in \mathbb{Z}$, $\{z, z+1, \ldots z+L\} \cap \Lambda \neq \emptyset$. In other words, Λ is relatively dense if it has only bounded gaps. Let X be a topological space and $T : X \to X$ a homeomorphism. A point $x \in X$ is almost periodic if for any neighbourhood U of x, the set $\{z \in \mathbb{Z} : T^z x \in U\}$ is relatively dense. For $x \in X$, the closure of the T-orbit of x will be denoted by $OC(x)$. Thus $OC(x) = \overline{\{T^z x : z \in \mathbb{Z}\}}$. A closed T-invariant set $F \subset X$ is minimal if it does not contain any non-empty, proper, closed T-invariant subset. The following proposition is easy to prove. (see Ellis)[2].

Proposition 1.1—Let X be a locally compact Hausdorff space and $T : X \to X$ a homeomorphism. A point $x \in X$ is almost periodic if and only if $OC(x)$ is compact and minimal.

Let $\mathbb{N} = \{1, 2, \ldots\}$, and $S = \{0,1\}$. We put the discrete topologies on $\mathbb{N}$ and S. Let $\Omega = \mathbb{N}^{\mathbb{Z}}$ and $X = S^{\mathbb{Z}}$ with the corresponding product topologies. We define the shifts $T' : \Omega \to \Omega$ and $T : X \to X$ as follows: $(T'\omega)(z) = \omega(z+1)$, for $\omega \in \Omega$ and $z \in \mathbb{Z}$; $(Tx)(z) = x(z+1)$, for $x \in X$ and $z \in \mathbb{Z}$. T' and T are homeomorphisms of Ω and X respectively.

A finite sequence $(a_1, \ldots, a_n)$ with $a_i \in S$ (or $\mathbb{N}$) is called a block in S (or $\mathbb{N}$); n is called the length of the block. The block $(a_1, \ldots, a_n)$ in S is said to occur in a point $x \in X$ at the place m if $x(m) = a_1, \ldots, x(m+n-1) = a_n$.

A block is said to occur in x if it occurs at some place.

For $x \in X$ and integers k and n with $n > 0$, $x[k : n]$ will denote the block $(x(k), \ldots, x(k+n-1))$. We similarly define $\omega[k ; n]$ for $\omega \in \Omega$. Also, S^* will denote the set of all blocks in S and for $b \in S^*$, $|b|$ will denote the length of the block b. Given a sequence of blocks $\{b_k\}_{k \in \mathbb{Z}}$ in S^*, there exists a unique $x \in X$ such that

$x[0; |b_0|] = b_0$

$$x\Big[\sum_{i=0}^{k-1} |b_i| ; |b_k|\Big] = b_k$$

and $[-\sum_{i=1}^{k} |b_{-i}| ; |b_{-k}|] = b_{-k}$

for all $k \geqslant 1$. We will denote this point by $\ldots b_{-1} \hat{b}_0 b_1 \ldots$

2. Construction of Orbit Closures with Unique Minimal Sets

Suppose that $x_0 \in X$ is an almost periodic point and $\theta : \mathbb{N} \to S^*$ a map such that the image of θ has infinitely many distinct blocks. We assume that θ is near x_0 in the sense that all but finitely many blocks in the image of θ occur in x_0. θ induces a map $\bar{\theta} : \Omega \to X$ as follows:

$$\bar{\theta}(\omega) = \ldots \theta(\omega(-1)) \widehat{\theta(\omega(0))} \theta(\omega(1)) \ldots, \text{ for } \omega \in \Omega.$$

Since we are interested only in the study of $OC(\bar{\theta}(\omega))$, we can assume, without loss of generality, that θ is injective. If $\omega \in \Omega$ is almost periodic and if $\omega \in F^{\mathbb{Z}}$ for some finite subset F of $\mathbb{N}$, then one can easily see that $\bar{\theta}(\omega)$ is also almost periodic and $OC(\bar{\theta}(\omega))$ in this case, can be analysed using the techniques in Jacobs and Keane[3] Henceforth we will consider only almost periodic points $\omega \in \Omega$ such that $\omega \notin F^{\mathbb{Z}}$ for any finite subset F of $\mathbb{N}$.

Proposition 2.1—Let $\bar{\theta} : \Omega \to X$ be as above and $\omega \in \Omega$, any almost periodic point such that $\omega \notin F^{\mathbb{Z}}$ for any finite subset F of $\mathbb{N}$. Then $OC(\bar{\theta}(\omega))$ contains $OC(x_0)$ as the unique minimal set.

Proof: It suffices to prove that for any block to occurring in x_0, the set $\Lambda_b = \{z \in \mathbb{Z} : \bar{\theta}(\omega)[z ; |b|] = b\}$ is relatively dense. Since x_0 is almost periodic, there exists an integer $L_b > 0$ such that b is a subblock of any block of length $> L_b$ occurring in x_0. Choose N such that for all $n > N$ the block $\theta(n)$ occurs in x_0 and $|\theta(n)| > L_b$. As $\omega \notin F^{\mathbb{Z}}$ for any finite $F \subset \mathbb{N}$, there exists an $n > N$ such that n occurs in ω. Furthermore, as ω is almost periodic, the set $\{z : \omega(z) > N\}$ is relatively dense. Let L denote the length of the maximum gap in $\{z : \omega(z) \geqslant N\}$. Then one can easily see that the length of any gap in Λ_b is atmost $(L' \max_{1 \leqslant i \leqslant N} |\theta(i)| + 2: L_b)$.

Thus Λ_b is relatively dense.

Let $\omega \in \Omega$ be an almost periodic point.

If $OC(\bar{\theta}(\omega)) = OC(x_0)$, then a necessary and sufficient condition for unique ergodicity of T on $OC(\bar{\theta}(\omega))$ is the unique ergodicity of T on $OC(x_0)$. Hence we can restrict our attention to almost periodic points $\omega \in \Omega$ such that $OC(x_0) \subsetneq OC\ \bar{\theta}(\omega))$. (This, in particular, implies that $\omega \notin F^{\mathbb{Z}}$ for any finite subset F of $\mathbb{N}$)- We prove:

Theorem 2.2—Let $\omega \in \Omega$ be an almost periodic point such that $OC(x_0) \subsetneq OC$

$(\bar{\theta}(\omega))$. $T : OC\,(\bar{\theta}(\omega)) \to OC\,(\bar{\theta}(\omega))$ is uniquely ergodic if and only if the following conditions are satisfied :

(i) $T : OC\,(x_0) \to OC\,(x_0)$ is uniquely ergodic.

(ii) $$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} |\,\theta(\omega(i+k))\,| = \infty$$

uniformly in k.

The next two sections will be devoted to the proof of the theorem.

3. Necessity

For convenience let us put, for $k \in \mathbb{Z}$ and $N = 1, 2, \ldots$

$$S_k\,(\omega\,;N) = \frac{1}{N} \sum_{i=0}^{N-1} |\,\theta(\omega(i+k))\,|\,.$$

The necessity of the condition (i) in (2.2) is clear. Now suppose that (i) is satisfied but the condition (ii) fails to hold. Then there exists an $\alpha < \infty$ such that given any non-negative integer n we can choose an integer $N > n$ and $k \in \mathbb{Z}$ such that $S_k\,(\omega\,;N) \leqslant \alpha$. Set $N_0 = 0$. Choose $N_1 > N_0$ and $k_1 \in \mathbb{Z}$ such that $S_{k_1}(\omega; N_1) \leqslant \alpha$.

Put

$$\Lambda_1 = \{z \in \mathbb{Z} : \omega\,[z-1; N_1+1] = \omega\,[k_1-1; N_1+1]\}.$$

Λ_1 is relatively dense. Let L_1 denote the length of the maximum gap in Λ_1. Choose $n_2 > 2\,(L_1 + N_1)$ and $p_2 \in \mathbb{Z}$ such that $S_{p_2}(\omega; n_2) \leqslant \alpha$. There exists an integer m_1 with $0 \leqslant m_1 < L_1 + N_1$ such that $k_2 = p_2 + m_1 \in \Lambda_1$.

Let $N_2 = n_2 - m_1$.

Then

$$S_{k_2}(\omega; N_2) = \frac{1}{N_2} \sum_{i=0}^{N_2-1} |\,\theta(\omega(i+k_2))\,|$$

$$\leqslant \frac{n_2}{n_2 - m_1}\, S_{p_2}(\omega\,; n_2)$$

$$\leqslant 2\,\alpha.$$

Also, $\omega\,[k_2-1; N_1+1] = \omega\,[k_1-1; N_1+1]$. Having defined $N_1, \ldots, N_l, k_1, \ldots, k_l$ such that

(a) $N_1 < N_2 < \ldots < N_l$.

(b) $S_{k_j}(\omega ; N_j) \leqslant 2\alpha, j = 1, \ldots l,$

and (c) $\omega[k_{j+1} - j ; N_j + j] = \omega[k_j - j; N_j + j]$ for all $j < l$

let $\wedge_l = \{z : \omega[z - l ; N_l + l] = \omega[k_l - l; N_l + l]\}$.

Then $\wedge_l$ is relatively dense. Let L_l denote the length of the maximum gap in $\wedge_l$. Choose $n_{l+1} > 2(L_l + N_l)$ and $p_{l+1} \in \mathbb{Z}$ such that $S_{p_{l+1}}(\omega; n_{l+1}) \leqslant \alpha$. There exists an integer m_l with $0 \leqslant m_l < L_l + N_l$ such that $k_{l+1} = p_{l+1} + m_l \in \wedge_l$; set $N_{l+1} = n_{l+1} - m_l$.

Then

$$S_{k_{l+1}}(\omega; N_{l+1}) \leqslant \frac{n_{l+1}}{n_{l+1} - m_l} S_{p_{l+1}}(\omega : n_{l+1})$$
$$\leqslant 2\alpha.$$

Also, since $k_{l+1} \in \wedge_l$, we have

$$\omega[k_{l+1} - l; N_l + l] = \omega[k_l - l; N_l + l].$$

This completes the inductive construction of $\{N_j ; k_j\}_{j=1}^{\infty}$, satisfying (a), (b) and (c) above. We now define $\omega' \in \Omega$ by setting

$$\omega'[-j; N_j + j] = \omega[k_j - j ; N_j + j], j = 1, 2, \ldots$$

Clearly ω' is almost periodic and $S_0(\omega'; N_j) \leqslant 2\alpha$ for all $j = 1, 2, \ldots$ Let us define

$$\sigma(\omega ; n) = \begin{cases} -\sum_{i=n}^{-1} |\theta(\omega(i))| & \text{if } n < 0 \\ 0 & n = 0 \\ \sum_{i=0}^{n-1} |\theta(\omega(i))| & \text{if } n > 0. \end{cases} \qquad \ldots(3.1)$$

Setting $x = \bar{\theta}(\omega)$, by our construction we immediately see that the sequence $\{T^{\sigma(\omega; k_j)} x\}_{j=1}^{\infty}$

converges to $\bar{\theta}(\omega')$. Since $OC(x) \neq OC(x_0)$, there exists a block B that occurs in x but not in x_0. As $x = \bar{\theta}(\omega)$, we can choose $n_1 \ldots, n_l \in \mathbb{N}$ such that the block $(n_1 \ldots n_l)$ occurs in ω and such that B is a subblock of $(\theta(n_1) \ldots \theta(n_l))$.

Note that the block $(n_1 \ldots, n_l)$ occurs in ω' also.

Let $\wedge = \{z : \omega'[z; l] = (n_1, \ldots, n_l)\}$. $\wedge$ is relatively dense and let L denote the the length of the maximum gap in $\wedge$. Suppose now that T is uniquely ergodic and let μ be the unique T-invariant probability measure on $OC(x)$. Consider the continuous function χ_B defined by

$$\chi_B(y) = \begin{cases} 1 & \text{if } y[0:|B|] = B \\ 0 & \text{otherwise}, \ y \in OC(x) \end{cases}$$

$$\int \chi_B \, d\mu = \lim_{j\to\infty} \frac{1}{\sigma(\omega'; N_j)} \sum_{i=0}^{\sigma(\omega'; N_j)-1} \chi_B(T^i x') \text{ where } x' = \overline{\theta}(\omega').$$

But for $j = 1, 2, \ldots$, we have

$$\frac{1}{\sigma(\omega'; N_j)} \sum_{i=0}^{\sigma(\omega'; N_j)-1} \chi_B(T^i x') \geqslant \frac{\frac{1}{N_j} \# (\wedge \cap [0, N_j))}{S_0(\omega'; N_j)}$$

$$\geqslant \frac{1}{2\alpha N_j} \# (\wedge \cap [0, N_j)).$$

Thus

$$\int \chi_B \, d\mu \geqslant \frac{1}{2\alpha L} > 0.$$

But as $OC(x_0)$ is the unique minimal subset of $OC(x)$, the unique invariant probability measure μ must be supported on $(OC(x_0)$; that is $\mu(OC(x_0)) = 1$ and consequently is, $\int \chi_B \, d\mu = 0$. The contradiction shows that the condition (ii) in the hypothesis is necessary.

4. Sufficiency

Suppose that (i) and (ii) of (2.2) hold. We will show that $T : OC(\overline{\theta}(\omega)) \to OC(\overline{\theta}(\omega))$ is uniquely ergodic. Let μ denote the unique T-invariant probability measure on $OC(x_0)$. For blocks b and b' in S^*, let $\rho_b(b')$ denote the relative frequency of the occurrence of b in b' i. e.

$$\rho_b(b') = \frac{\text{number of times } b \text{ occurs in } b'}{|b'|}.$$

Now let b be any block occurring x_0 and $\epsilon > 0$ arbitrary. Since μ is the unique T-invariant measure on $OC(x_0)$ we can choose L' such that for any block b' occurring in x_0 we have $|\rho_b(b') - \int \chi_b \, d\mu| < \epsilon$ if $|b'| > L'$. Choose M such that for all $m > M$, $\theta(m)$ occurs in x_0 and $|\theta(m)| > L'$.

Put $L = \max\limits_{1 \leqslant i \leqslant M} (|\theta(i)|, L')$.

For $N = 1, 2, \ldots$, let $a_N = \min\limits_{k} \sum\limits_{i=0}^{N-1} |\theta(\omega(i+k))|$.

By condition (ii) of (2.2) we can choose N_ϵ such that

$\frac{a_N}{N} > \frac{2L + |b|}{\epsilon}$, for all $N > N_\epsilon$. Setting $n_\epsilon = a_{N_\epsilon+1}$

we will show that for all $n \geqslant n_\epsilon$ and for all $k \in \mathbb{Z}$

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} \chi_b (T^{i+k} x) - \int \chi_b \, d\mu \right| < 4\epsilon \qquad \ldots(4.1)$$

where $x = \bar{\theta}(\omega)$.

Let $n \geqslant n_\epsilon$ and $k \in \mathbb{Z}$. Choose $k' \in \mathbb{Z}$ such that $\sigma(\omega; k') \leqslant k < \sigma(\omega; k' + 1)$ and let N be the smallest positive integer such that $k + n < \sigma(\omega; k' + N)$. If $N = 1$, then $x[k; n]$ is a subblock of $\theta(\omega(k'))$ and as $|\theta(\omega(k'))| > n > L$, the block $\theta(\omega(k'))$ occurs in x_0. Thus $x[k; n]$ also occurs in x_0 and as $n > L$, we have

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} \chi_b (T^{i+k} x) - \int \chi_b \, d\mu \right| = |\rho_b (x[k; n]) - \int \chi_b \, d\mu| < \epsilon.$$

Again if $N = 2$, then

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} \chi_b (T^{i+k} x) - \int \chi_b \, d\mu \right|$$

$$\leqslant \frac{1}{n} \{ |B_0| \, |\rho_b (B_0) - \int \chi_b \, d\mu| + |B_1| \, |\rho_b (B_1) - \int \chi_b \, d\mu| + |b| \}$$

where

$$B_0 = x[k; \sigma(\omega; k' + 1) - k]$$

and

$$B_1 = x[\sigma(\omega; k' + 1); k + n - \sigma(\omega; k' + 1)].$$

Now we have

$$|\rho_b (B_i) - \int \chi_b \, d\mu| < \epsilon \text{ if } |B_i| > L$$

and

$$|\rho_b (B_i) - \int \chi_b \, d\mu| \leqslant 2 \text{ if } |B_i| \leqslant L, \; i = 0, 1.$$

Thus

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} \chi_b (T^{i+k} x) - \int \chi_b \, d\mu \right| \leqslant \frac{2L + |b|}{n} + \epsilon < 2\epsilon$$

Now let $N > 2$. Put

$$B_0 = x[k; \sigma(\omega; k' + 1) - k]$$

$$B_i = x\,[\sigma(\omega;\, k' + i);\ \sigma(\omega;\, k' + i + 1) - \sigma(\omega;\, k' + i)],$$

$$\text{for } i = 1, \ldots, N - 1$$

and

$$B_N = x\,[\sigma(\omega;\, k' + N);\ k + n - \sigma(\omega;\, k' + N)].$$

$$\left|\frac{1}{n} \sum_{i=0}^{n-1} \chi_b(T^{i+k} x) - \int \chi_b\, d\mu\right.$$

$$< \frac{1}{n} \left\{ \sum_{j=0}^{N-1} |\, B_j \,| \,.\, |\, \rho_b(B_j) - \int \chi_b\ d\mu \,| + (N - 1)\ |\, b \,| \right\}.$$

If

$$|\, B_j \,| > L, \text{ then } |\, \rho_b(B_j) - \int \chi_j\, d\mu \,| < \epsilon$$

and

$$\text{if } |\, B_j \,| \leqslant L,\ |\, \rho_b(B_j) - \int \chi_b\, d\mu \,| \leqslant 2.$$

Thus,

$$\left|\frac{1}{n} \sum_{i=0}^{n-1} \chi_b(T^{i+k} x) - \int \chi_b\, d\mu\right|$$

$$\leqslant \frac{N(2L + |\, b \,|)}{n} + \epsilon.$$

Now choose N' such $a_{N'} \leqslant n < a_{N'+1}$.

Since $n \geqslant n_\epsilon = a_{N_\epsilon+1}$, we have $N' > N_\epsilon$. Also,

$$a_{N-2} \leqslant \sigma(\omega;\, k' + N - 1) - \sigma(\omega;\, k' + 1) < n < a_{N'+1}.$$

Thus $N < N' + 3$. Hence

$$\left|\frac{1}{n} \sum_{i=0}^{n-1} \chi_b(T^{i+k} x) - \int \chi_b\, d\mu\right| \leqslant \frac{(N' + 3)(2L + |\, b \,|)}{a_{N'}} + \epsilon$$

$$\leqslant 3\,\epsilon < 4\,\epsilon.$$

This establishes (4.1) and completes the proof of Theorem 2.2.

5. A Dichotomy

In this section using (2.2) we will prove the following theorem.

Theorem 5.1—$T : OC(\bar{\theta}(\omega)) \to OC(\bar{\theta}(\omega))$ is uniquely ergodic if and only if all points of $OC(\bar{\theta}(\omega))$ are quasi regular.

PROOF : The 'only if' part is well-known. Suppose now that $T : OC(\bar{\theta}(\omega)) \to OC(\bar{\theta}(\omega))$ is not uniquely ergodic. We have to show the existence of a non quasi-regular point. If $T : OC(x_0) \to OC(x_0)$ is itself not uniquely ergodic then as x_0 is almost periodic, $OC(x_0)$, and hence $OC(\bar{\theta}(\omega))$, has a non quasi-regular point. (See Prop. 3.2 of Katznelson and Weiss[4]). Now let $OC(x_0)$ have a unique T-invariant probability measure μ.

As $T : OC(\bar{\theta}(\omega)) \to OC(\bar{\theta}(\omega))$ is not uniquely ergodic, condition (ii) in (2.2) does not hold. Then there exists an $\alpha < \infty$, a sequence $\{N_j\}_{j=1}^{\infty}$ increasing to ∞, and an almost periodic point $\omega' \in \Omega$ such that $x' = \bar{\theta}(\omega') \in OC(\bar{\theta}(\omega))$ and $S_0(\omega' : N_j) \leqslant 2\alpha$ for all j. (See § 3 above). Now suppose that all points of $OC(\bar{\theta}(\omega))$ are quasi-regular. Then if μ' denotes the T-invariant measure for which x' is generic, the arguments in § 3 above show that $\mu' \neq \mu$. We prove :

Lemma—Let $G_\mu = \{y \in OC(x') : y \text{ is generic for } \mu\}$.

Then G_μ is dense in $OC(x')$.

PROOF : For each $n > 0$, we will find a point $y_n \in G_\mu$ such that the block $x'[\sigma(\omega'; -n); |\sigma(\omega'; -n)| + \sigma(\omega', n)]$ occurs in y_n. This is enough to conclude that G_μ is dense in $OC(x')$.

Let $n > 0$ be arbitary and let

$$\wedge_n = \{z \in \mathbb{Z} : \omega'[z; 2n+1] = \omega'[-n; 2n+1]\}.$$

$\wedge_n$ is relatively dense. Let L_n denote the length of the maximum gap in $\wedge_n$. Clearly, $\mathbb{Z} = \bigcup_{i=0}^{L_n - 1} \wedge_n + i$. Since $\omega' \notin F^{\mathbb{Z}}$ for any finite subset F of $\mathbb{N}$, there exists a least positive integer p such that $\omega'(\wedge_n + p)$ is infinite. For $k = 0, \ldots, p-1$, let $F_k = \omega'(\wedge_n + k)$. Each F_k is finite and

$F_k = \omega'(-n + k)$ for $k = 0, \ldots, 2n$. For $(i_0, \ldots, i_{p-1}) \in F_0 \times \ldots \times F_{p-1}$

put

$$\wedge(i_0, \ldots, i_{p-1}) = \{z + p : z \in \wedge_n, \omega'(z + k) = i_k, k = 0, \ldots, p-1\}.$$

Then since

$$\omega'(\wedge_n + p) \quad \bigcup_{(i_0, \ldots i_{p-1}) \in F_0 \times, \ldots \times F_{p-1}} \omega'(\wedge(i_0, \ldots i_{p-1}))$$

there exists $(l_0, \ldots, l_{p-1}) \in F_0 \times \ldots \times F_{p-1}$ such that $\omega'(\wedge(l_0, \ldots, l_{p-1}))$ is infinite. Choose a sequence $\{z_k + p\}_{k=1}^{\infty}$ in $\wedge(l_0, \ldots, l_{p-1})$ such that $\omega'(z_k + p) < \omega'(z_{k+1} + p)$ for all k. We can assume that the block $\theta(\omega'(z_k + p))$ occurs in x_0 for all k and hence we can find a sequence of integers $\{n_k\}$ such that

$$x_0 [n_k; \mid \theta(\omega'(z_k + p)) \mid] = \theta(\omega'(z_k + p)) \text{ for all } k.$$

We can assume that $\{T^{n_k} x_0\}$ converges to say $y_0 \in OC(x_0)$. Then if y_n is an accumulation point of the sequence $\{T^{\sigma(\omega'; z_k + p)} x'\}$ we have

$$\text{(a)} \quad y_n(z) = y_0(z) \text{ for all } z \geqslant 0$$

and

$$\text{(b)} \quad y_n [-K; \sigma(\omega'; n) + \mid \sigma(\omega'; -n) \mid]$$
$$= x[\sigma(\omega'; -n); \sigma(\omega'; n) + \mid \sigma(\omega'; -n) \mid]$$

where

$$K = \sum_{i=0}^{p-1} \mid \theta(l_i) \mid .$$

(a) implies that $y_n \in G_\mu$ and (b) shows that the block

$$x'[\sigma(\omega'; -n); \sigma(\omega'; n) + \mid \sigma(\omega'; -n) \mid] \text{ occurs in } y_n.$$

This completes the proof of the lemma.

Now we deduce a contradiction from our assumption that all points of $OC(\bar{\theta}(\omega))$ are quasi-regular. Since $\mu \neq \mu'$, there exists a complex-valued continuous function f defined on $OC(x')$ such that $\int f\, d\mu \neq \int f\, d\mu'$.

Let

$$\hat{f}(y) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N-1} f(T^i y), \; y \in OC(x').$$

By Baire's theorem, $\hat{f}$ has points of continuity. Let y_f be a point of continuity of $\hat{f}$. Since G_μ is dense in $OC(x')$, there exists a net $\{x_\beta\}$ in G_μ converging to y_f. Thus,

$$(1) \qquad \hat{f}(y_f) = \lim_\beta \hat{f}(x_\beta) = \int f\, d\mu \text{ as } x_\beta \in G_\mu \text{ for all } \beta.$$

On the other hand, as $y_f \in OC(x')$, there exists a sequence $\{T^{n_i} x'\}$ converging to y_f. Hence

$$(2) \qquad \hat{f}(y_f) = \lim \hat{f}(T^{n_i} x') = \int f\, d\mu'$$

as x' is generic for μ'. From (1) and (2) above, we have

$$\int f\, d\mu = \hat{f}(y_f) = \int f\, d\mu'$$

a contradiction. This completes the proof.

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