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TWO COMMUTATIVITY THEOREMS FOR SEMI PRIME RINGS

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In this paper the following two results have been proved (i) If R is a semi prime ring in which $(xy)^2 - yx^2y$ is central, for every x, y in R, then R is commutative (ii) Let n be a fixed positive integer and R be a semi prime ring in which $(xy)^n - yx$ is central, for every x, y in R. Then R is commutative.

We know that a group G satisfying $(xy)^2 = yx^2y$, for every x, y in G must be abelian. Recently we have obtained a ring theoretic analogue of the above group theoretic result for the ring with unity and provided an example to show that the result is not valid if the ring does not contain unity. In section 1 of the present paper, we generalize the above results for semi prime rings and provide an example which shows that the given theorem is not true in the case of arbitrary rings. Infact we prove the following :

Theorem 1—Let R be a semi prime ring in which $(xy)^2 - yx^2y$ is central, for every x, y in R. Then R is commutative.

Also authors[4] have discussed the commutativity for the rings in which for each pair of elements x, y, $(xy)^2 - xy$ is central. This study was motivated by the observation that a Boolean ring (satisfying $x^2 = x$) is necessarily commutative. In section 2 of the present paper we add to the above study and prove that :

Theorem 2 —Let n be a fixed positive integer and R be a semi prime ring in which $(xy)^n - yx$ is central, for every x, y in R. Then R is commutative.

Throughout the paper $Z(R)$ denotes the centre of an associative ring R and $[x, y] = xy - yx$ for any pair of elements x, y in R. In preparation for the proof of our theorems we first establish the following lemma.

Lemma—Suppose a prime ring R satisfies any one of the following polynomial identities :

(P_1) For all x, y in R, $(xy)^2 - yx^2 y \in Z(R)$.

(P_2) For all x, y in R and a fixed positive integer n,

$(xy)^n - yx \in Z(R)$.

Then R contains no nonzero zerodivisors.

PROOF : It suffices to show that R is reduced ring. Let a be an element of R such that $a^2 = 0$.

Using the hypothesis (P_1) of the lemma, we get $(ay)^2 \in Z(R)$, for all y in R and so in particular $(ay)^2 a = a(ay)^2$ which implies that $(ay)^3 = 0$, for all y in R. Thus by Lemma 1.1 of Herstein[2], we see that $aR = (0)$, the primeness of R forces that $a = 0$.

If R satisfies (P_2), then for any y in R, we have $\{(ay)^n - ya\} y = y \{(ay)^n - ya\}$. Now with $y = ay$, we have $ayaya = 0$ i. e. $(ay)^3 = 0$, for all y in R. Again $a = 0$ by Herstein[2].

SECTION 1

Proof of Theorem 1—Since R is semi prime in which $(xy)^2 - yx^2y$ is central, R is isomorphic to a subdirectsum of prime rings R_α each of which as a homomorphic image of R satisfies the hypothesis placed on R. Hence it is sufficient to prove the theorem in the case when R is prime in which $(xy)^2 - yx^2y$ is central.

First we assert that $Z(R) \neq (0)$. Assume on contrary that $Z(R) = (0)$. In that case $(xy)^2 = yx^2y$. With $y = x + y$ we get, $(xy - yx)\, x^2 = 0$. By Lemma (P_1), either $x^2 = 0$ or $xy - yx = 0$. If $x^2 = 0$ then $(x + y)^2 . x = 0$. Which implies that $xyx = 0$ i. e. $xRx = (0)$ and hence $x = 0$, since R is prime. But $x = 0$ also gives $xy - yx = 0$. Thus in both the cases $x \in Z(R) = (0)$, or $R = (0)$, a contradiction. Hence $Z(R) \neq (0)$.

Now let r be a nonzero element in $Z(R)$. R being prime, $ra \in Z(R)$ implies $a \in Z(R)$. On replacing y by $(y + r)$ in $(xy)^2 - yx^2y \in Z(R)$, we get $r(xyx - yx^2) \in Z(R)$. Which implies that $xyx - yx^2 \in Z(R)$. Now with $y = yx$, we have $(xyx - yx^2)\, x \in Z(R)$. Thus $x \in Z(R)$ unless $xyx - yx^2 = 0$. But $x \in Z(R)$ also gives $xyx - yx^2 = 0$. Hence in both the cases $(xy - yx)\, x = 0$. Again by Lemma (P_1), either $x = 0$ or $xy - yx = 0$ if $x = 0$, then also $xy - yx = 0$. Hence in both the cases $xy - yx = 0$ and R is commutative.

The following example shows that the above theorem is not true for arbitrary rings.

Example : Let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \Big/ \; a, b, c \text{ are integers} \right\}$.

It can be easily verified that $(xy)^2 - yx^2 y \in Z(R)$, for all x, y in R. However, R is not commutative.

SECTION 2

Proof of Theorem 2—Without loss of generality we may assume that R is a prime ring in which $(xy)^n - yx$ is central. By Lemma (P_2) R is reduced. Since prime reduced ring R is completely prime, according to Amitsur[1] R can be embedded in a division ring satisfying the same polynomial identity. Hence we can assume that R is a division ring in which $(xy)^n - yx \in Z(R)$. Now with $x = xy^{-1}$, we get $(xy^{-1}.y)^n - yxy^{-1} \in Z(R)$. Which implies that,

$$x^n y - yx = yx^n - y^2 xy^{-1}$$

i. e. $$[x^n, y]\, y - y\, [x, y] = 0. \qquad \ldots(1)$$

Replace y by $(x + y)$ in (1), to get

$$[x^n, y]\, x - x\, [x, y] = 0. \qquad \ldots(2)$$

Again with $y = x^2 + y$ in (1), we have

$$[x^n, y]\, x^2 - x^2\, [x, y] = 0 \qquad \ldots(3)$$

Multiply equation (2) by x from the right, to get

$$[x^n, y]\, x^2 - x\, [x, y]\, x = 0. \qquad \ldots(4)$$

On combining (3) and (4), we get $x^2 [x, y] - x [x, y] x = 0$. Which implies that $x [x, y] - [x, y] x = 0$, for all $x, y \in R$. If ch $R \neq 2$, then result follows by sub-lemma of Herstein[2], if ch $R = 2$ then we have $x^2 \in Z(R)$ i.e. $[x, y] \in Z(R)$. Now i. e. $[x, y] \in Z(R)$. Now with $x = xy$ we get $[x, y]\, y \in Z(R)$. Thus $y \in Z(R)$ unless $[x, y] = 0$. But if $y \in Z(R)$ then also $[x, y] = 0$. Thus in both the cases R is commutative.

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