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CERTAIN CLASSES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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We introduce the classes $S_n(A, B)$ and $K_n(A, B)$ of analytic functions with negative coefficients by using the nth order Ruscheweyh derivative. For these classes, in this paper, we find the coefficients estimate, extreme points and support points. We prove distortion theorems. We also consider the modified Hadamard product of functions in $S_n(A, B)$ and in $K_n(A, B)$.

INTRODUCTION

Let A denote the class of functions $f(z)$ analytic in the unit disc $E = \{z : |z| < 1\}$. Let V denote the subclass of A consisting functions normalised by $f(0) = 0$ and $f'(0) = 1$. The Hadamard product $(f * g)(z)$ of two functions $f(z) = \sum_{m=0}^{\infty} a_m z^m$ and $g(z) = \sum_{m=0}^{\infty} b_m z^m$ in A is given by $(f * g)(z) = \sum_{m=0}^{\infty} a_m b_m z^m$.

Let $D^{\alpha} f(z) = \frac{z}{(1-z)^{\alpha+1}} * f(z)$, $(\alpha \geqslant -1)$. Ruscheweyh[6] observed that $D^n f(z) = z(z^{n-1} f(z))^{(n)}/n!$ when $n \in N \cup \{0\}$, where $N = \{1, 2, 3, \ldots\}$. This symbol $D^n f(z)$, $(n \in N \cup \{0\})$ was called the nth Ruscheweyh derivative of $f(z)$ by Al-Amiri[1].

Let T be the subclass of V consisting functions of the form $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geqslant 0$. Let $S_n(A, B)$ denote the class of functions $f \in T$ such that

$$\frac{D^{n+1} f(z)}{D^n f(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \quad -1 \leqslant A < B \leqslant 1$$

where $n \in N \cup \{0\}$ and $\omega(z) \in H = \{\omega$ analytic, $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in E\}$. Let $K_n(A, B)$ denote the class of functions $f \in T$ such that $zf'(z) \in S_n(A, B)$.

In this paper, we find the coefficient inequalities and determine the extreme points, support points and radius of convexity. We prove distortion theorems. We also consider the modified Hadamard product of functions in $S_n(A, B)$ and in $K_n(A, B)$.

The results obtained by Owa[5] can be deduced from the corresponding results for the class $S_n(A, B)$ by taking $A = 0$ and $B = 1$. The class $S_0(\beta(2\gamma - 1), \beta)$, $(0 \leqslant \gamma < 1, 0 < \beta \leqslant 1)$, reduces to the class investigated by Gupta and Jain[4]. The classes $S_0(2\gamma - 1, 1)$ and $K_0(2\gamma - 1, 1)$, $0 \leqslant \gamma < 1$, reduce to the classes studied by Silverman[7].

2. Coefficients Inequalities

Theorem 1—Let $f \in T$. Then $f \in S_n(A, B)$ if and only if

$$\sum_{m=2}^{\infty} \frac{(n + m - 1)!\, c_m\, a_m}{(n + 1)!\,(m - 1)!} \leqslant B - A \qquad \ldots(2.1)$$

where

$$c_m = (B + 1)(n + m) - (A + 1)(n + 1).$$

Proof : Suppose $f \in S_n(A, B)$. Then

$$\frac{D^{n+1} f(z)}{D^n f(z)} = \frac{1 + A\,\omega(z)}{1 + B\,\omega(z)}, \quad -1 \leqslant A < B \leqslant 1,\ \omega \in H,\ z \in E.$$

From this we get

$$\omega(z) = \frac{D^n f(z) - D^{n+1} f(z)}{BD^{n+1} f(z) - AD^n f(z)}$$

and $|\omega(z)| < 1$ implies

$$|\omega(z)| = \left| \frac{\sum_{m=2}^{\infty} \frac{(n + m - 1)!}{(n + 1)!\,(m - 1)} [(n + m) - (n + 1)]\, a_m z^m}{(B - A) z - \sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!\,(m-1)!} [B(n+m) - A(n+1)] a_m z^m} \right| < 1. \qquad \ldots(2.2)$$

Therefore,

$$\operatorname{Re} \left\{ \frac{\sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!\,(m-1)!} [(n+m) - (n + 1)]\, a_m z^{m-1}}{(B-A) - \sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!\,(m-1)!} [B(n+m) - A(n+1)] a_m z^{m-1}} \right\} < 1. \qquad \ldots(2.3)$$

We consider real values of z and take $z = r$ with $0 \leqslant r < 1$. Then, for $r = 0$, denominator of (2.3) is positive and so it is positive for all r with $0 < r < 1$, since $\omega(z)$ is analytic for $|z| < 1$. Then (2.3) gives

$$\sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!\,(m-1)!}\, c_m\, a_m\, r^{m-1} < B - A. \qquad \ldots(2.4)$$

Letting $r \to 1$, we get (2.1).

Conversely, suppose $f \in T$ and satisfies (2.1). For $|z| = r$, $0 \leqslant r < 1$, we have (2.4) by (2.1), since $r^{m-1} < 1$. So we have

$$\left| \sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!\,(m-1)!} [(n+m) - (n+1)]\, a_m\, z^{m-1} \right|$$

$$\leqslant \sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!\,(m-1)!} [(n+m) - (n+1)]\, a_m\, r^{m-1}$$

$$< B - A - \sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!\,(m-1)!} [B(n+m) - A(n+1)]\, a_m r^{m-1}$$

$$\leqslant \left| B - A - \sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!\,(m-1)!} [B(n+m) - A(n+1)\, a_m\, z^{m-1} \right|$$

which gives (2.2) and hence follows that

$$\frac{D^{n+1} f(z)}{D^n f(z)} = \frac{1 + A\,\omega(z)}{1 + B\,\omega(z)},\ \omega \in H,\ z \in E,\ -1 \leqslant A < B \leqslant 1.$$

That is, $f \in S_n(A, B)$.

Corollary 1—If $f \in T$ is in $S_n(A, B)$, then

$$a_m \leqslant \frac{(n+1)!\,(m-1)!\,(B-A)}{(n+m-1)!\,c_m}$$

for $m \geqslant 2$. The equality holds for the functions f given by

$$f(z) = z - \frac{(n+1)!\,(m-1)!\,(B-A)}{(n+m-1)!\,c_m} z^m,\ z \in E.$$

Theorem 2—Let $f \in T$. Then $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geqslant 0$ is in $K_n(A, B)$ if and only if

$$\sum_{m=2}^{\infty} \frac{(n+m-1)!\, m c_m a_m}{(n+1)!\,(m-1)!} \leqslant B - A.$$

3. Closoure Theorems

Theorem 3—The class $S_n(A, B)$ is closed under convex linear combination.

Proof: Let $f, g \in S_n(A, B)$ and let $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $g(z) = z - \sum_{m=2}^{\infty} b_m z^m$. For η such that $0 \leqslant \eta \leqslant 1$, it suffices to show that the function h defined by $h(z) = (1-\eta) f(z) + \eta g(z)$, $z \in E$ belongs to $S_n(A, B)$. Now $h(z) = z - \sum_{m=2}^{\infty} [(1-\eta) a_m + \eta b_m] z^m$. Applying Theorem 1 to $f, g \in S_n(A, B)$, we have

$$\sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!\,(m-1)!} c_m [(1-\eta) a_m + \eta b_m]$$

$$= (1-\eta) \sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!\,(m-1)!} c_m a_m + \eta \sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!\,(m-1)!} c_m b_m$$

$$\leqslant (1-\eta)(B-A) + \eta = (B-A).$$

This implies $h \in S_n(A, B)$.

Theorem 4—The class $K_n(A, B)$ is closed under convex linear combination.

Theorem 5—Define

$$f_1(z) = z \text{ and } f_m(z) = z - \frac{(n+1)!\,(m-1)!\,(B-A)}{(n+m-1)!\, c_m} z^m,$$

$$m = 2, 3, 4, \ldots, z \in E. \tag{3.1}$$

Then $f \in S_n(A, B)$ if and only if f can be expressed as

$$f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z), \text{ where } \mu_m \geqslant 0 \text{ and } \sum_{m=1}^{\infty} \mu_m = 1.$$

Proof: If $f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z)$ with $\sum_{m=1}^{\infty} \mu_m = 1$, $\mu_m \geqslant 0$, then

$$\sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!\,(m-1)!}\,\mu_m \frac{(n+1)!\,(m-1)!\,(B-A)}{(n+m-1)!\,c_m}$$

$$= \sum_{m=2}^{\infty} \mu_m (B-A) = (1-\mu_1)(B-A) \leqslant B-A$$

and hence $f \in S_n(A, B)$.

Conversely, let

$$f(z) = z - \sum_{m=2}^{\infty} a_m z^m \in S_n(A, B).$$

Define

$$\mu_m = \frac{(n+m-1)!\,c_m\,a_m}{(n+1)!\,(m-1)!\,(B-A)},\ m = 2, 3,\ldots,$$

and define

$\mu_1 = 1 - \sum_{m=2}^{\infty} \mu_m$. From Theorem 1, we have $\sum_{m=2}^{\infty} \mu_m \leqslant 1$ and so $\mu_1 \geqslant 0$. Since $\mu_m f_m(z) = \mu_m z - a_m z^m$,

$$\sum_{m=1}^{\infty} \mu_m f_m(z) = z - \sum_{m=2}^{\infty} a_m z^m = f(z).$$

Theorem 6—Define

$$f_1(z) = z \text{ and } f_m(z) = z - \frac{(n+1)!\,(m-1)!\,(B-A)}{(n+m-1)!\,mc_m} z^m,$$

$$m = 2, 3,\ldots,\ z \in E.$$

Then $f \in K_n(A, B)$ if and only if f can be expressed as

$$f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z), \text{ where } \mu_m \geqslant 0 \text{ and } \sum_{m=1}^{\infty} \mu_m = 1.$$

4. Distortion Theorems

Theorem 7—Let the function $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geqslant 0$ be in the class $S_n(A, B)$. Then

$$|z| - (B-A)|z|^2/c_2 \leqslant |f(z)| \leqslant |z| + (B-A)|z|^2/c_2 \ldots (4.1)$$

and

$$1 - 2(B - A)\,|z|/c_2 \leqslant |f'(z)| \leqslant 1 + 2(B - A)\,|z|/c_2 \qquad \ldots(4.2)$$

for $z \in E$. The results are sharp for the function $f(z) = z - (B - A)\,z^2/c_2$.

PROOF :

$$|f(z)| = |z - \sum_{m=2}^{\infty} a_m z^m| \leqslant |z| + |z|^2 \sum_{m=2}^{\infty} a_m$$

and

$$|f(z)| \geqslant |z| - |z|^2 \sum_{m=2}^{\infty} a_m.$$

Since $\frac{(n+m-1)!}{(n+1)!\,(m-1)!}\,c_m$ is an increasing function of $m \geqslant 2$, and $f(z) \in S_n(A, B)$, by Theorem 1, we have

$$\frac{(n+1)!\,c_2}{(n+1)!\,1!} \sum_{m=2}^{\infty} a_m \leqslant \sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!\,(m-1)!}\,c_m a_m \leqslant B - A$$

that is,

$$\sum_{m=2}^{\infty} a_m \leqslant \frac{B - A}{c_2}. \qquad \ldots(4.3)$$

From (4.3), we get (4.1)

$$|f'(z)| = |1 - \sum_{m=2}^{\infty} m a_m z^{m-1}| \leqslant 1 + |z| \sum_{m=2}^{\infty} m a_m$$

and

$$|f'(z)| \geqslant 1 - |z| \sum_{m=2}^{\infty} m a_m.$$

Since $\frac{(n+m-1)!\,c_m}{(n+1)!\,m!}$ is an increasing function of $m \geqslant 2$ and $\frac{(n+m-1)!\,c_m}{(n+1)!\,(m+1)!}$ $= \frac{(n+m-1)!\,mc_m}{(n+1)!\,m!}$, by Theorem 1, we have

$$\frac{(n+1)!\,c_2}{(n+1)\,1!\,2}\sum_{m=2}^{\infty} ma_m \leqslant \sum_{m=2}^{\infty}\frac{(n+m-1)!\,c_m\,a_m}{(n+1)!\,(m-1)!} \leqslant B-A$$

that is,

$$\sum_{m=2}^{\infty} ma_m \leqslant 2\,(B-A)/c_2. \qquad \ldots(4.4)$$

From (4.4), we get (4.2). Further for the function $f(z) = z - (B-A)\,z^2/c_2$, we can see that the results of the theorem are sharp.

Corollary 2—$f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geqslant 0$ be in the class $S_n(A, B)$. Then $f(z)$ is included in a disk with its center at the origin and radius r given by $r = (c_2 + B - A)/c_2$ and $f'(z)$ is included in a disk with its center at the origin and radius r_1 given by $r_1 = [c_2 + 2\,(B-A)]/c_2$.

Theorem 8—Let the function $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geqslant 0$ be in the class $K_n(A, B)$. Then

$$|z| - (B-A)\,|z|^2/2c_2 \leqslant |f(z)| \leqslant |z| + (B-A)\,|z|^2/2c_2$$

and

$$1 - (B-A)\,|z|/c_2 \leqslant |f'(z)| \leqslant 1 + (B-A)\,|z|/c_2$$

for $z \in E$. The results are sharp for the function

$$f(z) = z - (B-A)\,z^2/2c_2.$$

5. Radius of Starlikeness and Convexity

Lemma 1[7]—A function $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geqslant 0$ is starlike univalent in E if and only if $\sum_{m=2}^{\infty} ma_m \leqslant 1$.

Lemma 2[7]—A function $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geqslant 0$ is convex univalent in E if and only if $\sum_{m=2}^{\infty} m^2 a_m \leqslant 1$.

Theorem 9—(i) $f \in S_n(A, B)$ implies f is starlike univalent. (ii) $f \in K_n(A, B)$ implies f is convex univalent.

PROOF : Let $f \in S_n(A, B)$. Then, by (4.4), we have

$$\sum_{m=2}^{\infty} m a_m \leqslant 2(B - A)/c_2 \leqslant 1.$$

This implies, by Lemma 1[7], f is starlike univalent.

Let $f \in K_n(A, B)$. Since $\frac{(n+m-1)!\, c_m}{(n+1)!\, m!}$ is an increasing function of $m \geqslant 2$ and $f(z) \in K_n(A, B)$, by Theorem 2, we have

$$\frac{(n+1)!\, c_2}{(n+1)!\, 2!} \sum_{m=2}^{\infty} m^2 a_m \leqslant \sum_{m=2}^{\infty} \frac{(n+m-1)!\, m\, c_m a_m}{(n+1)!\, (m-1)!} \leqslant B - A$$

that is,

$$\sum_{m=2}^{\infty} m^2 a_m \leqslant 2(B - A)/c_2 \leqslant 1.$$

Hence, by Lemma 2[7], f is convex univalent.

Theorem 10—If $f(z) \in S_n(A, B)$, then f is convex for $|z| < r_2$, where

$$r_2 = \inf_m \left[\frac{(n+m-1)!\, c_m}{m(n+1)!\, m!\, (B - A)} \right]^{\frac{1}{m-1}}.$$

PROOF : It is sufficient to show that $\left| \frac{zf''}{f'} \right| \leqslant 1$ for $|z| < r_2$.

$$\left| \frac{zf''}{f'} \right| = \left| \frac{-\sum_{m=2}^{\infty} m(m-1) a_m z^{m-1}}{1 - \sum_{m=2}^{\infty} m a_m z^{m-1}} \right| \leqslant \frac{\sum_{m=2}^{\infty} m(m-1) a_m |z|^{m-1}}{\left| 1 - \sum_{m=2}^{\infty} m a_m z^{m-1} \right|}. \quad \ldots(5.1)$$

Consider the values of z for which

$$|z| \leqslant \inf_m \left[\frac{(n+m-1)!\, c_m}{m(n+1)!\, m!\, (B - A)} \right]^{\frac{1}{m-1}}$$

that is,

$$|z|^{m-1} \leqslant \frac{(n+m-1)!\,c_m}{m\,(n+1)!\,m!\,(B-A)}$$

holds. Then

$$\sum_{m=2}^{\infty} ma_m |z|^{m-1} \leqslant \sum_{m=2}^{\infty} \frac{(n+m-1)!\,c_m}{(n+1)!\,m!\,(B-A)}\, a_m.$$

Now

$$\sum_{m=2}^{\infty} ma_m |z|^{m-1} < 1$$

provided

$$\sum_{m=2}^{\infty} \frac{(n+m-1)!\,c_m}{(n+1)!\,m!\,(B-A)}\, a_m < 1. \qquad \ldots(5.2)$$

Now if

$$f \in S_n(A, B), \sum_{m=2}^{\infty} \frac{(n+m-1)!\,c_m\,a_m}{(n+1)!\,(m-1)!\,(B-A)} \leqslant 1$$

and since

$$\sum_{m=2}^{\infty} \frac{(n+m-1)!\,c_m\,a_m}{(n+1)!\,m!\,(B-A)} < \sum_{m=2}^{\infty} \frac{(n+m-1)!\,c_m\,a_m}{(n+1)!\,(m-1)!\,(B-A)}$$

(5.2) holds. Therefore we can rewrite the denominator of the right hand side of inequality (5.1), for the considered values of z, using the fact that

$$1 > \sum_{m=2}^{\infty} ma_m |z|^{m-1}.$$

Thus

$$\left|\frac{z f''(z)}{f'(z)}\right| \leqslant \frac{\sum_{m=2}^{\infty} m(m-1)\,a_m |z|^{m-1}}{1 - \sum_{m=2}^{\infty} ma_m |z|^{m-1}} \leqslant 1$$

if

$$\sum_{m=2}^{\infty} m^2 a_m \mid z \mid^{m-1} \leqslant 1. \qquad \ldots(5.3)$$

By Theorem 1 and Lemma 1, we need only find values of r_2 for which

$$m^2 \mid z \mid^{m-1} \leqslant \frac{(n+m-1)!\, c_m}{(n+1)!\,(m-1)!\,(B-A)} \quad (m = 2.3,\ldots,).$$

The above inequality will be true when $\mid z \mid < r_2$. This completes the proof.

6. Support Points

The family A is a locally convex topological space with the topology of uniform convergence on compact subset of E. Let S denote the subfamily of V consisting of functions f which are univalent in E. It is known that S is a compact subset of A. A continuous linear functional on A is defined as a complex-valued functional defined on A which is linear and continuous. Let P be a convex subfamily of S. The set of extreme points of P will be denoted by Ext (P).

A point $f \in P$ is called a support point of P if there exists a continuous linear functional J on A such that Re $J(f) \geqslant$ Re $J(g)$ for all $g \in P$, and Re J is non constant on P. The set of support points of P will be denoted by Supp (P). We need the following lemmas.

Lemma 2[3]—Each continuous linear functional J on A is given by a sequence $\{b_m\}$ of complex numbers satisfying

$$\overline{\lim} \mid b_m \mid^{\frac{1}{m}} < 1 \text{ and } J(f) = \sum_{m=0}^{\infty} a_m b_m$$

where

$$f(z) = \sum_{m=0}^{\infty} a_m b_m \in A.$$

Let P be a subfamily of S whose set of extreme points is countable, suppose f_0 is a support point of P and J be a corresponding continuous linear functional. Let

$$Q_J = \{f \in P : \text{Re } J(f) = \text{Re } J(f_0)\}.$$

Lemma 4[3]—Q_J is convex, Ext $(Q_J) \subset$ Ext (P) and

$$Q_J = \{f \in P : f = \sum_{i=1}^{\infty} \delta_i f_i,\ \delta_i \geqslant 0 \ \sum_{i=1}^{\infty} \delta_i = 1$$

$$\text{and } f_i \in \text{Ext } (Q_J),\ i = 1, 2,\ldots\}.$$

Theorem 11—The set of support points of $S_n(A, B)$ is given by

$$\text{Supp}\,(S_n(A,B)) = \Big\{ f \in S_n(A,B) : f(z) = z - \sum_{m=2}^{\infty} \frac{(n+1)!\,(m+1)!\,(B-A)\delta_m}{(n+m-1)!\,c_m} z_m$$

$$\delta_m \geqslant 0,\ \sum_{m=2}^{\infty} \delta_m \leqslant 1 \text{ and } \delta_j = 0 \text{ for some } i \Big\}.$$

PROOF: By Theorem 3 and Theorem 9, $S_n(A, B)$ is a convex subfamily of S. Define f_0 by

$$f_0(z) = z - \sum_{m=2}^{\infty} \frac{(n+1)!\,(m-1)!\,(B-A)\,\delta_m}{(n+m-1)!\,c_m} z_m$$

where

$$\sum_{m=2}^{\infty} \delta_m \leqslant 1,\ \delta_m \geqslant 0 \text{ and } \delta_j = 0 \text{ for some } j < 1.$$

$f_0 \in S_n(A, B)$, since (2.1) is satisfied for f_0. Ler J be the sequence $\{b_m\}$ of complex numbers where $b_m = 0$, $m > 1$, $m \neq j$ and $b_1 = b_j = 1$. Then J is a continuous linear functional on A by Lemma 3. Now $J(f_0) = 1$ and for

$$f(z) = z - \sum_{m=2}^{\infty} a_m z^m \in S_n(A, B),\ J(f) = 1 - a_j \leqslant 1\ (a_j \geqslant 0).$$

Hence f_0 is a support point of $S_n(A, B)$.

Conversely, suppose that f_0 is a support point of $S_n(A, B)$ and that J is a continuous linear functional on $S_n(A, B)$ given by sequence $\{b_m\}$. Since $\lceil \text{Re } J$ is continuous and linear on $S_n(A, B)$, by the Krein-Milman theorem there exists f_m, an extreme point of $S_n(A, B)$, such that $\text{Re } J(f_0) = \text{Max}\{\text{Re } J(f) : f \in S_n(A, B)\} = \text{Re } J(f_m)$. Let

$$Q_J = \{f_m : \text{Re } J(f_0) = \text{Re } J(f_m), f_m \in \text{Ext}\,(S_n A, B))\}.$$

If $Q_J = \text{Ext}\,(S_n(A, B))$, then Re J must be constant on $S_n(A, B)$. Since J is not constant on $S_n(A, B)$, there exists a j such that $\text{Re } J(f_j) < \text{Re } J(f_0)$. From this we have

$$\text{Ext}\,(Q_J) \subset \{f_m : f_m \in \text{Ext}\,(S_n(A, B)), m = 1, 2, \ldots, m \neq j\}.$$

Hence, by Lemma 4, $f_0(z) = \sum_{m=2}^{\infty} \delta_m f_m$, where $\delta_m \geqslant 0$, $\Sigma \delta_m = 1$ and $f_m \in \text{Ext}(Q_J)$, $m = 1, 2, \ldots$. That is

$$f_0(z) = z - \sum_{\substack{m=1 \\ m \neq j}}^{\infty} \frac{(n+1)!\,(m+1)!\,(B-A)\,\delta_m}{(n+m-1)!\,c_m} z_m$$

which completes the proof of the theorem.

Remark : This theorem reduces to Theorem 2.1[3].

Theorem 12—The set of support points of $K_n(A, B)$ is given by

$$\text{Supp}(K_n(A, B)) = \left\{ f \in K_n(A, B) : f(z) = z - \sum_{m=2}^{\infty} \frac{(n+1)!\,(m-1)!\,(B, A)\,\delta_m}{(n+m-1)!\,mc_m} z^m, \right.$$

$$\left. \delta_m \geqslant 0, \sum_{m=2}^{\infty} \delta_m \leqslant 1 \text{ and } \delta_j = 0 \text{ for some } j \right\}.$$

7. Modified Hadamard Product

Let $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geqslant 0$ and $g(z) = z - \sum_{m=2}^{\infty} b_m z^m$, $b_m \geqslant 0$. Further let $(f *_1 g)(z)$ denote the modified Hadamard product of $f(z)$ and $g(z)$, that is

$$(f *_1 (z) = z - \sum_{m=2}^{\infty} a_m b_m z^m.$$

Theorem 13—Let $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geqslant 0$ be in $S_{n_1}(A, B)$ and let $g(z) = z - \sum_{m=2}^{\infty} b_m z^m$, $b_m \geqslant 0$ be in $S_{n_2}(A, B)$. Then the modified Hadamard product $(f *_1 g)(z)$ belongs to $S_{n_1}(A, B) \cap S_{n_2}(A, B)$.

Proof : Since $g(z) \in S_{n_2}(A, B)$ we have from (4.3) that

$$b_m < \frac{B - A}{c_{2,2}}, \qquad \ldots(7.1)$$

where

$$c_{m,i} = (B + 1)(n_i + m) - (A + 1)(n_i + 1),\ i = 1, 2.$$

From Theorem 1, since $f(z) \in S_{n_1}(A, B)$, we have

$$\sum_{m=2}^{\infty} \frac{(n_1 + m - 1)!\, c_{m,1}\, a_m}{(n_1 + 1)!\,(m - 1)!} \leqslant B - A. \qquad \ldots(7.2)$$

Now, from (7.1) and (7.2),

$$\sum_{m=2}^{\infty} \frac{(n_1 + m - 1)!\, c_{m,1}\, a_m\, b_m}{(n_1 + 1)!\,(m + 1)!}$$

$$\leqslant \frac{B - A}{c_{2,2}} \sum_{m=2}^{\infty} \frac{(n_1 + m - 1)!\, c_{m,1}\, a_m}{(n_1 + 1)!\,(m - 1)!}$$

$$\leqslant \frac{(B - A)^2}{c_{2,2}} \leqslant B - A.$$

Since $[(B - A)/c_{2,2}] \leqslant 1$. Hence $(f *_1 g)(z) \in S_{n_1}(A, B)$. Interchanging n_1 and n_2 by each other in the above, we get $(f *_1 g)(z) \in S_{n_2}(A, B)$. Hence the theorem.

Theorem 14—Let $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geqslant 0$ be in $K_{n_1}(A, B)$ and let $g(z) = z - \sum_{m=2}^{\infty} b_m z^m$, $b_m \geqslant 0$ be in $K_{n_2}(A, B)$. Then the modified Hadamard product $(f *_1 g)(z)$ belongs to $K_{n_1}(A, B) \cap K_{n_2}(A, B)$.

Theorem 15—Let $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geqslant 0$ be in $K_{n_1}(A, B)$ and let $g(z) = z - \sum_{m=2}^{\infty} b_m z^m$, $b_m \geqslant 0$ be in $S_{n_2}(A, B)$. Then the modified Hadamard product $(f *_1 g)(z)$ belongs to $K_{n_1}(A, B) \cap S_{n_2}(A, B)$.

PROOF : Since $g(z) \in S_{n_2}(A, B)$, we have

$$b_m \leqslant \frac{B - A}{c_{2,2}}.$$

Using Theorem 2, we have

$$\sum_{m=2}^{\infty} \frac{(n_1 + m - 1)!\, m\, c_{m,1}\, a_m\, b_m}{(n_1 + 1)!\,(m - 1)!}$$

$$\leqslant \frac{B - A}{c_{2,2}} \sum_{m=2}^{\infty} \frac{(n_1 + m - 1)!\, m\, c_{m,1}\, a_m}{(n_1 + 1)!\,(m - 1)!}$$

$$\leqslant \frac{(B - A)^2}{c_{2,2}} \leqslant B - A.$$

This implies $(f *_1 g)(z) \in K_{n_1}(A, B)$. Similarly, we can prove $(f *_1 g)(z) \in S_{n_2}(A, B)$.

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