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A ROTATING AXISYMMETRIC EXPANDING UNIVERSE WITH AN ELECTROMAGNETIC FIELD

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An exact axially symmetric solution of Einstein Maxwell equations corresponding to a mixture of a rotating perfect fluid, a source-free electromagnetic field and pure radiation field is derived. The solution discussed by Vaidya and Patel [*Gen. Rel. Grav.* **16** (1984), 355-64] has been derived as a particular case. The details of the solution are also discussed.

1. INTRODUCTION

Vaidya and Patel[1] have given a new physical interpretation to the closed Robertson-Walker space-time. They have seen that the geometry of this space time can also accomodate the gravitational field of a homogeneous rotating fluid in the presence of a universal sourcefree electromagnetic field and a pure radiation field. The rotation of the stream lines of the perfect fluid and the radiation density were linked with the electromagnetic field. When the electromagnetic field disappears, the rotation and the radiation density vanish.

The usual Robertson-Walker universe have spherical symmetry. It is also well-known that these universes are conformally flat. Bondasenko and Kohashkin[2] have obtained conformaly flat axisymmetric generalizations of Robertson-Walker universes. The principal aim of the present investigation is to obtain such axially symmetric generalization of the solution discussed by Vaidya and Patel[1].

Vaidya and Patel[1] have shown that the usual metric of the Einstein universe,

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - \frac{(xdx + ydy + zdz)^2}{R_0^2 - (x^2 + y^2 + z^2)} \qquad ...(1.1)$$

can be transformed to the form

$$ds^2 = 2\,(du + R_0 \sin^2 \tfrac{1}{2} \propto d\beta)\, dt - (du + R_0 \sin^2 \tfrac{1}{2} \propto d\beta)^2 - \tfrac{1}{4} R_0^2\,(d\alpha^2 + \sin^2 \propto d\beta^2) \qquad ...(1.2)$$

where, R_0 is a constant.

We also know that the metric of the closed Robertson-Walker universe is conformal to (1.2), the conformal factor being a function of time t.

2. Equations of Structure

We take Einstein universe described by the metric (1.2). We consider a metric conformal to (1.2) in the form

$$ds^2 = (1/F^2)\,[2\,(du + R_0 \sin^2 \tfrac{1}{2}\,\alpha\, d\beta)\,dt - (du + R_0 \sin^2 \tfrac{1}{2}\,\alpha\, d\beta)^2$$

$$- \tfrac{1}{4}\,R_0^2\,(d\alpha^2 + \sin^2 \alpha\, d\beta^2)] \qquad \ldots(2.1)$$

where F is a function of the co-ordinates α, t and r. The co-ordinate r is defined by $r = t - u$.

For the computation of Ricci-tensor for the metric (2.1) we shall employ the Cartan's approach of exterior differential forms. This technique of differential forms is standard now and therefore we shall not go into the details.

Let us introduce the following basic 1-forms in the Riemannian manifold defined by the metric (2.1).

$$\theta^1 = (1/F)\,(du + R_0 \sin^2 \tfrac{1}{2}\,\alpha\, d\beta),\ \theta^2 = (R_0/2F)\,d\alpha$$

$$\theta^3 = (R_0/2F) \sin \alpha\, d\beta,\ \theta^4 = (1/F)\,dt - (1/2)\,\theta^1. \qquad \ldots(2.2)$$

The metric (2.1) can now be expressed as

$$ds^2 = 2\theta^1\,\theta^4 - (\theta^2)^2 - (\theta^3)^2 = g_{(ab)}\,\theta^a\,\theta^b. \qquad \ldots(2.3)$$

Using (2.2) it is easy to compute $d\theta^a$ and Cartan's first structure equation $d\theta^a = -\,w_b^a \wedge \theta^b$ will then give the connection 1-forms w_b^a.

A straight forward computation gives

$$w_1^1 = -\,w_4^4 = \tfrac{1}{2}\,(F_r - F_t)\,\theta^1 + (F_r + F_t)\,\theta^4,$$

$$w_2^1 = w_4^2 = -\,(2F_\alpha/R_0)\,\theta^1 - (F_r + F_t)\,\theta^2 + (F/R_0)\,\theta^3$$

$$w_3^1 = w_4^3 = -\,F_r \tan \tfrac{1}{2}\,\alpha\,\theta^1 - (F/R_0)\,\theta^2 - (F_r + F_t)\,\theta^3$$

$$w_1^2 = w_2^4 = \tfrac{1}{2}\,(F_r + F_t)\,\theta^2 - (F/2R_0)\,\theta^3 - (2F/R_0)\,\theta^4$$

$$w_1^3 = w_3^4 = (F/2R_0)\,\theta^2 + \tfrac{1}{2}\,(F_r - F_t)\,\theta^3 - F_r \tan \tfrac{1}{2}\,\alpha\,\theta^4$$

$$w_3^2 \equiv -\,w_2^3 = -(F/2R_0)\,\theta^1 - F_r \tan \tfrac{1}{2}\,\alpha\,\theta^2 + (2/R_0)\,(F_\alpha - F \cot \alpha)\,\theta^3$$

$$+ (F/R_0)\,\theta^4 \qquad \ldots(2.4)$$

with $w_{ab} + w_{ba} = 0$, and a suffix denoting partial derivative

(e.g. $F_\alpha = \partial F/\partial\alpha$, $F_r = \partial F/\partial r$).

From the second equation of structure,

$$dw^a_b + w^a_c \wedge w^c_b = \tfrac{1}{2} R^a_{bcd}\ \theta^c \wedge \theta^d$$

we can obtained the tetrad components R^a_{bcd} of the curvature tensor. For the sake of brevity, they are not listed here. Using these components it is easy to obtain the tetrad components $R_{(ab)} = R^c_{abc}$ of the Ricci-tensor for the metric (2.1).

The explicit expressions for $R_{(ab)}$ are listed below :

$$R_{(12)} = (F/R_0)\,[2\,(F_{r\alpha} - F_{t\alpha}) + F_r \tan\tfrac{1}{2}\alpha]$$

$$R_{(13)} = (F/R_0^2)\,[-R_0^2 \tan\tfrac{1}{2}\alpha\,(F_{tr} - F_{rr}) - 2F_\alpha]$$

$$R_{(24)} = -(2F/R_0)\,[2\,(F_{\alpha r} + F_{\alpha t}) + F_r \tan\tfrac{1}{2}\alpha]$$

$$R_{(34)} = (2F/R_0^2)\,[-R_0^2 \tan\tfrac{1}{2}\alpha\,(F_{rr} + F_{rt}) + 2F_\alpha]$$

$$R_{(23)} = (2F \tan(\alpha/2)\,R_0)\,[2F_{\alpha r} + F_t \tan\tfrac{1}{2}\alpha]$$

$$R_{(11)} = F\,[F_{tr} - 1/2\,(F_{rr} + F_{tt}) - (F/2R_0^2)]$$

$$R_{(22)} = (4/R_0^2)\,[3F_\alpha^2 - 3FF_{\alpha\alpha} - FF_\alpha \cot\alpha - \tfrac{1}{2}F^2] \qquad \ldots(2.5)$$

$$+\ 3\,(F_r^2 \sec^2\tfrac{1}{2}\alpha - F_t^2) + F\,(F_{tt} - F_{rr}\sec^2\tfrac{1}{2}\alpha)$$

$$R_{(33)} = (4/R_0^2)\,[3F_\alpha^2 - 3FF_\alpha \cot\alpha - FF_{\alpha\alpha} - \tfrac{1}{2}F^2$$

$$+\ 3\,(F_r^2 \sec^2\tfrac{1}{2}\alpha - F_t^2) + F\,(F_{tt} - F_{rr} - 3F_{rr}\tan^2\tfrac{1}{2}\alpha)]$$

$$R_{(44)} = -2F\,[F_{tt} + F_{rr} + 2F_{rt} + (F/R_0^2)]$$

$$R_{(14)} = (4/R_0^2)\,[F_\alpha F \cot\alpha + FF_{\alpha\alpha} - 3F_\alpha^2 + \tfrac{1}{4}F^2]$$

$$+\ F\,(F_{rr}\tan^2\tfrac{1}{2}\alpha - 2F_{tt} + 2F_{rr}) + 3\,(F_t^2 - F_r^2\sec^2\tfrac{1}{2}\alpha).$$

3. The Field Equations

We shall use the following field equations,

$$R_{ik} - (1/2)\,g_{ik}\,R = -8\pi[(p+\rho)\,v_i\,v_k - pg_{ik} + E_{ik}]$$

$$-8\pi\sigma\,w_i\,w_k - \Lambda\,g_{ik} \qquad \ldots(3.1)$$

$$E_{ik} = -g^{lm} F_{il} F_{km} + \tfrac{1}{4} g_{ik} F_{lm} F^{lm} \quad \ldots(3.2)$$

$$F_{ik} = A_{i,k} - A_{k,i} \quad \ldots(3.3)$$

$$F^{ik}_{\ ;k} = 0. \quad \ldots(3.4)$$

The symbols occuring in the above field equations have this usual meanings. The velocity vectors v^i and w^i satisfy the relations.

$$v_i v^i = 1,\ v_i w^i = 1,\ w_i w^i = 0. \quad \ldots(3.5)$$

We choose the 4-potential A_i as

$$A = (A(t), 0, A(t) R_0 \sin^2 \tfrac{1}{2}\alpha, 0). \quad \ldots(3.6)$$

Here and in what follows the co-ordinates are taken as, $u = x^1$, $\alpha = x^2$, $\beta = x^3$ and $t = x^4$. From (3.3) and (3.6) one can determine the electromagnetic field tensor F_{ik}. Using these F_{ik} we have verified that the Maxwell's equations (3.4) are satisfied if,

$$A_t^2 + \frac{4A^2}{R_0^2} = \text{constant} = b^2 \text{ (say)} \quad \ldots(3.7)$$

With the aid of (3.2), we have determined the electromagnetic energy tensors E_{ik}. They are given by

$$\begin{aligned} &E_{12} = E_{23} = E_{24} = E_{44} = 0 \\ &-E_{11} = -R_0 E_{13}/\sin^2 \tfrac{1}{2}\alpha = E_{14} = E_{22} = E_{34}/R_0 \sin^2 \tfrac{1}{2}\alpha \\ &\qquad = E_{33}/R_0 \sin^2 \tfrac{1}{2}\alpha \cos\alpha = \tfrac{1}{2} b^2 F^2. \end{aligned} \quad \ldots(3.8)$$

The tetrad components $E_{(ab)}$ of E_{ik} are obtained from

$$E_{(ab)} = e^i_{(a)} e^k_{(b)} E_{ik},\ e^i_{(a)} \theta^a = dx^i.$$

A straight forward calculation gives,

$$E_{(14)} = E_{(22)} = E_{(33)} = \tfrac{1}{2} b^2 F^4. \quad \ldots(3.9)$$

The remaining components of $E_{(ab)}$ are zero. The field equation (3.1) can be expressed in the tetrad form as

$$\begin{aligned} R_{(ab)} = &-8\pi[(p + \rho) v_{(a)} v_{(b)} - \tfrac{1}{2}(\rho - p) g_{(ab)} + E_{(ab)}] \\ &- 8\pi\sigma w_{(a)} w_{(b)} + \Lambda g_{(ab)} \end{aligned} \quad \ldots(3.10)$$

For the metric (2.1) we take

$$v_{(a)} = (1/2\lambda, 0, 0, \lambda),\ w_{(a)} = (1/\lambda, 0, 0, 0) \quad \ldots(3.11)$$

where, λ is a parameter to be determined from the field equations.

The results (3.9), (3.10) and (3.11) imply the following relations.

$$R_{(23)} = R_{(12)} = R_{(13)} = R_{(24)} = R_{(34)} = 0,\ R_{(22)} = R_{(33)} \qquad ...(3.12)$$

$$R_{(11)} = -2\pi\,(p + \rho)/\lambda^2 - 8\pi\,\sigma/\lambda^2 \qquad ...(3.13)$$

$$R_{(22)} = -\Lambda - 4\pi\,(\rho - p) - 8\pi\,E_{(22)} \qquad ...(3.14)$$

$$R_{(14)} = \Lambda - 8\pi\,p - 8\pi\,E_{(14)} \qquad ...(3.15)$$

$$R_{(44)} = -8\pi\,(p + \rho)\,\lambda^2 \qquad ...(3.16)$$

where $R_{(ab)}$ and $E_{(ab)}$ are given by (2.5) and (3.9) respectively.

4. A Rotating Axisymmetric Expanding Universe

It is painless to verify that the eqns. (3.12) admit the solution,

$$F = B\,(r)\cos\tfrac{1}{2}\,\alpha + g\,(t);\ d^2B/dr^2 + B/R_0^2 = 0 \qquad ...(4.1)$$

where $g\,(t)$ is an arbitrary function of time t.

We take

$$B = KR_0 \sin\,(r/R_0). \qquad ...(4.2)$$

The four equations (3.13), (3.14), (3.15) and (3.16) are sufficient to determine the four parameters p, ρ, λ^2 and σ. They given by

$$8\pi\,(p + \rho) = -16\pi\,b^2\,F^4 + 2F\,(g_{tt} + g/R_0^2) \qquad ...(4.3)$$

$$8\pi p = \Lambda - 4\pi b^2\,F^4 + 3K^2 + 2F\,(g_{tt} + g/R_0^2) - 3\,(g_t^2 + g^2/R_0^2) \qquad ...(4.4)$$

$$\lambda^2 = (g_{tt} + g/R_0^2)/[(g_{tt} + g/R_0^2) - 8\pi\,b^2\,F^4] \qquad ...(4.5)$$

$$16\pi\,\sigma = F\,(\lambda^2 - 1)\,[(g_{tt} + g/R_0^2) + 8\pi\,b^2\,F^4] \qquad ...(4.6)$$

where, F is given by (4.1) and (4.2). The final form of the metric of our solution is,

$$ds^2 = [KR_0 \sin\,(r/R_0)\cos\,\alpha/2 + g\,(t)]^{-2}\,[2\,(du + R_0 \sin^2\tfrac{1}{2}\,\alpha\,d\beta)\,dt - (du + R_0 \sin^2\tfrac{1}{2}\,\alpha\,d\beta)^2 - (1/4)\,R_0^2\,(d\,\alpha^2 + \sin^2\alpha\,d\beta^2)]. \qquad ...(4.7)$$

If $K = 0$, then applying the transformation $(1/g^2)\,dt = dx$ and writing $g^2 = R_0/R(x)$ we get the solution discussed by Vaidya and Patel[3]. The metric in this case becomes,

$$ds^2 = 2\,(du + R_0 \sin^2\tfrac{1}{2}\,\alpha\,d\beta)\,dx - (R/R_0)\,(du + R_0 \sin^2\tfrac{1}{2}\,\alpha\,d\beta)^2 - (1/4)\,RR_0\,(d\alpha^2 + \sin^2\alpha\,d\beta^2) \qquad ...(4.8)$$

Vaidya and Patel[1] have shown that the metric (4.8) represents rotating expanding homogeneous universe with a source-free electromagnetic field. Therefore the metric (4.7) represents a rotating axisymmetric expanding universe with an electromagnetic field.

If $g(t) =$ constant then the metric (4.7) describes a stationary generalization of Einstein universe with a source-free electromagnetic field.

If we set $b = 0$ in the above discussion, the electromagnetic field disappears. Also $b = 0$ implies $\lambda^2 = 1$ and $\sigma = 0$.

The pressure and density are given by

$$8\pi p = \Lambda + 3K^2 + 2F(g_{tt} + g/R_0^2) - 3(g_t^2 + g^2/R_0^2) \quad \ldots(4.9)$$

$$8\pi(p + \rho) = 2F(g_{tt} + g/R_0^2). \quad \ldots(4.10)$$

It this case we recover a conformally flat axisymmetric universe discussed by Bondasenko and Kohashkin[2].

We have also verified that when $b = 0$ the rotation of the flow vector v^i vanishes. The vanishing of the constant b does not make any change in the metric. Thus the metric (4.7) admits two distinct types of sources. Therefore two different physica interpretations can be given to the metric (4.7). Firstly the metric (4.7) describes a gravitational field due to a rotating perfect fluid in the presence of a source-free electromagnetic field and a pure radiation field. Secondly the same metric (4.7) describes a gravitational field due to a non-rotating perfect fluid distribution. The second physical interpretation of (4.7) has been already discussed by Bondasenko and Kohashkin[2]. The first physical interpretation of (4.7) discussed by us is a new one.

Further if $g_{tt} + g/R_0^2 = 0$, then $g_t^2 + g^2 R_0^2 =$ constant $= n^2$ (say)

In this case $p = \rho = 0$ and $\Lambda = 3(n^2 - k^2)$

Thus in the case $b = 0$, $g_{tt} + g/R_0^2 = 0$ we have a solution of $R_{ik} = \Lambda g_{ik}$, where $\Lambda = 3(n^2 - k^2)$

If $n^2 > k^2$, Λ is positive and one gets deSitter universe.

If $n^2 < k^2$, Λ is negative and we get anti deSitter universe.

If $n^2 = k^2$, $\Lambda = 0$ and $R_{ik} = 0$.

But we have verified that in this case the curvature tensor vanishes.

Thus, $b = 0$, $g_{tt} + g/R_0^2 = 0$ and $n = k$ implies the flat space-time.

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