Indian J. pure appl. Math., 17 (10) : 1231-1241, October 1986

NUMERICAL SOLUTION OF THE FLOW OF A SECOND-ORDER FLUID THROUGH A CHANNEL WITH POROUS WALLS UNDER A TRANSVERSE MAGNETIC FIELD

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(*Received* 2 *December* 1985)

The problem of steady flow of an electrically conducting incompressible non-Newtonian second-order fluid through a channel with porous walls has been solved by finite difference technique using the Newton Raphson method. The behaviour of the velocity profile for different values of second-order parameter, Hartmann and Reynolds numbers is shown graphically. The results are compared with those obtained by the regular perturbation technique.

1. INTRODUCTION

The flow of an electrically conducting fluid between porous boundaries is of practical interest in problems of gaseous diffusion etc. The problem of steady laminar flow of an incompressible viscous fluid in a two dimensional channel with uniformly porous walls has been studied by Berman[1], Sellars[2] and Yuan[3]. Thereafter Terrill and Shrestha[4] discussed this problem for the case when the walls are of different permeability and studied the effects of magnetic field when the fluid is electrically conducting[5]. The problem of flow of a second-order fluid with heat transfer in a channel with porous walls has been considered by Agarwal[6].

The present paper is devoted to the study of steady flow of a non-Newtonian second-order fluid through a channel with porous walls under a transverse magnetic field by finite difference technique. The governing differential equations and boundary conditions are replaced by algebraic equations in unknown variables by using finite-difference approximations for the derivatives. The algebraic equations thus obtained are solved by the most popular Newton-Raphson method. The results obtained by regular perturbation technique are used as the first approximation to the numerical solution and compared with those obtained by finite difference method for small Reynolds number. The second-order effects on the velocity profile are illustrated graphically for different values of the Hartmann and Reynolds numbers. It is worth mentioning here that Newton-Raphson method being fast converging is capable of solving a quite difficult set of non-linear equations to a good degree of accuracy in few iterations.

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2. Formulation of the Problem

The constitutive equation of an incompressible second-order fluid suggested by Coleman and Noll[7] can be written as :

$$\tau_{ij} = -p\delta_{ij} + 2\mu_1 d_{ij} + 2\mu_2 e_{ij} + 4\mu_3 c_{ij} \quad \ldots(1)$$

where

$$\left.\begin{aligned} d_{ij} &= \frac{1}{2}(v_{i,j} + v_{j,i}) \\ e_{ij} &= \frac{1}{2}(a_{i,j} + a_{j,i}) + v^m_{,i}\, v_{m,j} \\ c_{ij} &= d^m_i\, d_{mj} \end{aligned}\right\} \quad \ldots(2)$$

p is the hydro-static pressure; τ_{ij} is the stress-tensor; v_i and a_i are the velocity and the acceleration vector and μ_1, μ_2, μ_3 represent material constants whose values are given by $\mu_1 = 18.5$, $\mu_2 = -0.2$ and $\mu_3 = 1.0$ (all expressed C. G. S. units) for a 5.46 percent solution of poly-iso-butylene in cetane at 30°C as suggested by Markovitz[8].

We consider the steady flow of an incompressible second-order fluid in a channel, of width $2h$, consisting of two porous walls (coinciding with the planes $y = \pm h$) of equal permeability. A constant magnetic field H_0 is applied normal to the axis of the channel. The induced magnetic field has been neglected in the flow since the magnetic Reynolds number is small. A uniform suction V is applied to both the walls of the channel. Let us choose the origin of a rectangular co-ordinate system in the middle of the channel with x and y-axes respectively in a plane parallel and perpendicular to the channel walls. Let u and v be the componets of the velocity in x and y-directions respectively.

Following Terrill and Shrestha[4] a stream function ψ can be defined as

$$\psi(x, \zeta) = (hU - Vx) f(\zeta) \quad \ldots(3)$$

where U is the entrance velocity and $\zeta\ (= y/h)$ is the dimensionless distance while $2h$ is the distance between the channel walls. Accordingly,

$$\left.\begin{aligned} u(x, \zeta) &= \left(U - \frac{Vx}{h}\right) f'(\zeta) \\ v(\zeta) &= V f(\zeta) \end{aligned}\right\} \quad \ldots(4)$$

where dash denotes differentiation with respect to ζ. The expression (4) suggests that u is a function of x and ζ, while v is a function of ζ only. Using this fact, the constitutive equation (1) the equation of continuity and momentum equations can be written as :

$$\frac{\partial u}{\partial x} + \frac{1}{h}\frac{\partial v}{\partial \zeta} = 0 \qquad \ldots(5)$$

$$u\frac{\partial u}{\partial x} + \frac{v}{h}\frac{\partial u}{\partial \zeta} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu_1}{h^2}\frac{\partial^2 u}{\partial \zeta^2} + \nu_2\left[\frac{1}{h^2}\frac{\partial^2}{\partial \zeta^2}\left(u\frac{\partial u}{\partial x} + \frac{v}{h}\frac{\partial v}{\partial \zeta}\right) + \frac{2}{h^2}\frac{\partial}{\partial \zeta}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial \zeta}\right)\right]$$

$$+ \frac{3\nu}{h^2}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \zeta}\right)^2 - \frac{\mu_e^2 H_0^2\,\sigma}{\rho}u \qquad \ldots(6)$$

and

$$v\frac{\partial v}{\partial \zeta} = -\frac{1}{\rho}\frac{\partial p}{\partial \zeta} + \frac{\nu_1}{h}\frac{\partial^2 v}{\partial \zeta^2} + \nu_2\left[\frac{2}{h}\frac{\partial^2}{\partial \zeta^2}\left(\frac{v}{h}\frac{\partial v}{\partial \zeta}\right) + 2\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial \zeta}\right) + \frac{4}{h^2}\left(\frac{\partial u}{\partial \zeta}\frac{\partial^2 u}{\partial \zeta^2} + \frac{\partial v}{\partial \zeta}\frac{\partial^2 v}{\partial \zeta^2}\right) - \frac{\partial^2}{\partial x\,\partial \zeta}\left(u\frac{\partial u}{\partial x} + \frac{v}{h}\frac{\partial u}{\partial \zeta}\right)\right]$$

$$+ \frac{\nu_3}{h^2}\left[4\frac{\partial}{\partial \zeta}\left(\frac{\partial v}{\partial \zeta}\right)^2 + \frac{\partial}{\partial \zeta}\left(\frac{\partial u}{\partial \zeta}\right)^2\right]. \qquad \ldots(7)$$

The boundary conditions are,

$$u(x, \pm 1) = 0, \left(\frac{\partial u}{\partial \zeta}\right)_{\zeta=0} = 0$$

$$v(x, 0) = 0, v(x, 1) = V \text{ and } v(x, -1) = -V. \qquad \ldots(8)$$

Using (4) in equations (6) and (7) are eliminating p, we get

$$f^{iv} + R(f'f'' - ff''') + \tau_1(ff^v - f'f^{iv}) + S^2 f'' = 0 \qquad \ldots(9)$$

where $R\,(= Vh/\nu_1)$ is the suction Reynolds number, $\tau_1\left(= \frac{\nu_2 V}{h\nu_1}\right)$ is an elastico-viscous parameter governing the effects of elastico-viscosity of the fluid and $S\,[= \mu_e H_0 h (\sigma/\mu_1)^{\frac{1}{2}}]$ is the Hartmann number. It can be concluded from eqn. (9) that the two dimensional flows are independent of cross-viscosity of the fluid. The pressure alone gets modified due to the this property of the fluid[9].

3. Solution of the Problem

i Perturbation Solution

Assuming the relationships $\tau_1 = -R\tilde{\tau}_1\,(\tilde{\tau}_1 \geqslant 0)$ and $S^2 = RS_1^2$, eqn. (9) becomes :

$$f^{iv} + (f'f'' - ff''') - R\tilde{\tau}_1(ff^v - f'f^{iv}) + RS_1^2 f'' = 0. \qquad \ldots(10)$$

Equation (10) has to be solved under the boundary conditions :

$$f(0) = f'(1) = f''(0) = 0, f(1) = 1. \qquad ...(11)$$

For small values of the suction Reynolds number R, we can develop a regular perturbation scheme for solving eqn. (10) by expanding f in powers of R. Substituting

$$f(\zeta) = \sum_{n=0}^{\infty} R^n f_n(\zeta) \qquad ...(12)$$

in eqn. (10) and equating like powers of R on the two sides of the resulting equations we obtain the following set of equations :

$$f_0^{iv} = 0$$

$$f_1^{iv} + f_0' f_0'' - f_0 f_0''' - \tilde{\tau}_1 \left(f_0 f_0^{v} - f_0' f_0^{iv} \right) + S_1^2 f_0'' = 0$$

$$f_2^{iv} + f_0' f_1'' + f_1' f_0'' - f_0 f_1''' - f_1 f_0''' - \tilde{\tau}_1 \left(f_0 f_1^{v} + f_1 f_0^{v} - f_0' f_1^{iv} - f_1' f_0^{iv} \right) + S_1^2 f_1'' = 0$$

$$f_3^{iv} + f_0' f_2'' + f_2' f_0'' + f_1' f_1'' - f_0 f_2''' - f_2 f_0''' - f_1 f_1''' - \tilde{\tau}_1 \left(f_0 f_2^{v} + f_2 f_0^{v} + f_1 f_1^{v} - f_0' f_2^{iv} - f_2' f_0^{iv} - f_1' f_1^{iv} \right) + S_1^2 f_2'' = 0. \qquad ...(13)$$

The boundary conditions (11) in terms of f_0, f_1 etc. can be written as :

$$f_n(0) = f_n'(1) = f_n''(0) = 0, \text{ for all } n;$$

$$f_0(1) = 1, f_n(1) = 0, \text{ for } n \geq 1. \qquad ...(14)$$

The solution of the equations (13) subjected to the boundary conditions (14) is given as follows :

$$f_0(\zeta) = \frac{1}{2}(3\zeta - \zeta^3) \qquad ...(15a)$$

$$f_1(\zeta) = -\frac{1}{280}(\zeta^7 - 3\zeta^3 + 2\zeta) + \frac{S_1^2}{40}(\zeta^5 - 2\zeta^3 + \zeta) \qquad ...(15b)$$

$$f_2(\zeta) = \frac{1}{1293600}(14\zeta^{11} - 385\zeta^9 + 198\zeta^7 + 876\zeta^3 - 703\zeta)$$

$$- \frac{\tilde{\tau}_1}{280}\{3\zeta^7 - 9\zeta^3 + 6\zeta - S_1^2(\zeta^7 - 3\zeta^3 + 2\zeta)\}$$

$$+ S_1^2\left\{\frac{1}{100800}(15\zeta^9 + 108\zeta^7 - 54\zeta^5 - 276\zeta^3 + 207\zeta)\right.$$

$$\left. - \frac{S_1^2}{8400}(5\zeta^7 - 21\zeta^5 + 27\zeta^3 - 11\zeta)\right\} \quad \ldots(15c)$$

$$f_3(\zeta) = -\frac{1}{1293600}\left(\frac{133}{780}\zeta^{15} - \frac{49}{13}\zeta^{13} + \frac{162}{5}\zeta^{11} - \frac{55}{5}\zeta^9 - \frac{291}{8}\zeta^7\right.$$

$$\left. - \frac{98461}{2730}\zeta^3 + \frac{16376}{455}\zeta\right) + \tilde{\tau}_1\left[\frac{1}{1293600}\left(-\frac{210}{13}\zeta^{13} + 315\zeta^{11}\right.\right.$$

$$\left. - 3465\zeta^9 + 1254\zeta^7 + \frac{112059}{13}\zeta^3 - \frac{87201}{13}\zeta\right)$$

$$- \frac{S_1^2}{100800}\left\{\frac{45}{11}\zeta^{11} - \frac{54155}{378}\zeta^9 - \frac{1296}{7}\zeta^7 + 162\zeta^5\right.$$

$$+ \frac{2191355}{2079}\zeta^3 - \frac{1111189}{1386}\zeta + S_1^2(15\zeta^9 + 108\zeta^7 - 54\zeta^5$$

$$\left. - 276\zeta^3 + 207\zeta)\right\}$$

$$\left. - \frac{\tilde{\tau}_1}{280}\{9\zeta^7 - 27\zeta^3 + 18\zeta - S_1^2(3\zeta^7 - 9\zeta^3 + 6\zeta)\}\right]$$

$$- S_1^2\left[\frac{1}{7761600}\left(\frac{70}{13}\zeta^{13} - \frac{987}{10}\zeta^{11} - 605\zeta^9 + \frac{2739}{5}\zeta^7\right.\right.$$

$$\left. + \frac{1314}{5}\zeta^5 + \frac{92585}{130}\zeta^3 - \frac{53591}{65}\zeta\right)$$

$$+ S_1^2\left\{\frac{1}{100800}\left(\frac{3}{11}\zeta^{11} + \frac{9}{2}\zeta^9 - \frac{492}{35}\zeta^7 - \frac{69}{5}\zeta^5\right.\right.$$

$$+ \frac{19407}{385}\zeta^3 - \frac{21039}{770}\zeta\Big) - \frac{S_1^2}{8400}\left(\frac{5}{72}\zeta^9 - \frac{1}{2}\zeta^7\right.$$

$$\left.\left.\left. - \frac{27}{20}\zeta^5 - \frac{266}{180}\zeta^3 + \frac{67}{120}\zeta\right)\right\}\right]. \quad \ldots(15d)$$

If $S_1 = 0$ and $\tilde{\tau}_1 = 0$, $S_1 = 0$, the results are in good agreement with those obtained by Agrawal[6] and Terrill[10] respectively.

(ii) Numerical Solution

To solve the highly non-linear equation (9) by finite difference technique, we first convert the differential equation (9) into corresponding finite difference equation by employing the following difference quotients :

$$f'_j = \frac{f_{j+1} - f_{j-1}}{2h} \qquad \ldots(16a)$$

$$f''_j = \frac{f_{j+1} - 2f_j + f_{j-1}}{h^2} \qquad \ldots(16b)$$

$$f'''_j = \frac{-f_{j-2} - 2f_{j-1} - 2f_{j+1} + f_{j+2}}{2h^3} \qquad \ldots(16c)$$

$$f^{iv}_j = \frac{f_{j-2} - 4f_{j-1} + 6f_j - 4f_{j+1} + f_{j+2}}{h^4} \qquad \ldots(16d)$$

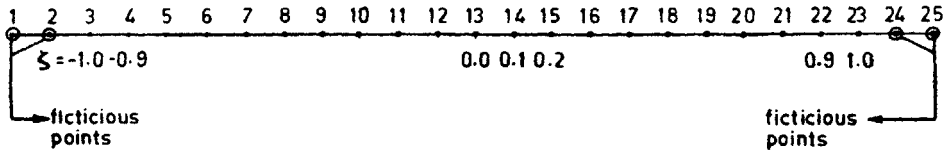
$$f^{v}_j = \frac{-f_{j-3} + 4f_{j-2} - 5f_{j-1} + 5f_{j+1} - 4f_{j+2} + f_{j+3}}{2h^5} \qquad \ldots(16e)$$

where h denotes the interval between equidistant values of the independent variable ζ. Thus the finite difference equation corresponding to (9) is

$$\begin{aligned} F_j(f_{j-3}, f_{j-2}, f_{j-1}, f_j, f_{j+1}, f_{j+2}, f_{j+3}) \\ = 2h(f_{j-2} - 4f_{j-1} + 6f_j - 4f_{j+1} + f_{j+2}) + Rh^2(-f^2_{j-1} + f_{j-2}f_j \\ + f^2_{j+1} - f_j f_{j+2}) + \tau_1(-f_{j-3}f_j + f_{j-2}f_{j-1} + 4f_{j-2}f_j - f_{j-2}f_{j+1} \\ - 4f^2_{j-1} + f_{j-1}f_j + f_{j-1}f_{j+2} - f_j f_{j+1} \\ + 4f^2_{j+1} - 4f_{j+2}f_j - f_{j+2}f_{j+1} + f_{j+3}f_j) \\ + 2S^2h^3(f_{j-1} - 2f_j + f_{j+1}) = 0 \end{aligned} \qquad \ldots(17)$$

where $4 \leqslant j \leqslant 22$.

We divide ζ [— 1, 1] into twenty equal parts each of length $h = 0.10$ with mesh points 3 to 23. Mesh points 1, 2, 24 and 25 are taken as fictitious points.



The boundary conditions (8) can be rewritten as

$$f_2 = f_4, f_3 = -1, f_{13} = 0, f_{12} = -f_{14}, f_{22} = f_{24}, f_{23} = 1. \qquad \ldots(18)$$

The finite difference equation (17) represents a system of 19 non-linear equations in 19 variables

$$f_1, f_4, f_5, f_6 \ldots\ldots, f_{12}, f_{15}, f_{16}, f_{17}, \ldots\ldots, f_{21}, f_{22}, f_{25}.$$

The first approximation to f_j (value of f at $\zeta = \zeta_j$) may be obtained from eqn. (15). Let the exact solutions be $\bar{f}_4, \bar{f}_5, \ldots, \bar{f}_{22}$, then the differences Δf_j are calculated from

$$\bar{f}_j = f_j + \Delta f_j, \; 4 \leqslant j \leqslant 22. \qquad \ldots(19)$$

Expanding F_j by means of Taylor series expansion and neglecting the second and higher powers of small quantities, we get

$$\bar{F}_j = F_j + \left(\frac{\partial F_j}{\partial f_{j-3}} \Delta f_{j-3} + \frac{\partial F_j}{\partial f_{j-2}} \Delta f_{j-2} + \frac{\partial F_j}{\partial f_{j-1}} \Delta f_{j-1} + \frac{\partial F_j}{\partial f_j} \Delta f_j \right.$$
$$\left. + \frac{\partial F_j}{\partial f_{j+1}} \Delta f_{j+1} + \frac{\partial F_j}{\partial f_{j+2}} \Delta f_{j+2} + \frac{\partial F_j}{\partial f_{j+3}} \Delta f_{j+3} \right) = 0. \qquad \ldots(20)$$

Equation (20) can be represented as

$$J \begin{bmatrix} \Delta f_1 \\ \Delta f_4 \\ \Delta f_5 \\ \Delta f_6 \\ \vdots \\ \Delta f_{12} \\ \Delta f_{15} \\ \Delta f_{16} \\ \vdots \\ \Delta f_{21} \\ \Delta f_{22} \\ \Delta f_{25} \end{bmatrix} + \begin{bmatrix} F_1 \\ F_4 \\ F_5 \\ F_6 \\ \vdots \\ F_{12} \\ F_{15} \\ F_{16} \\ \vdots \\ F_{21} \\ F_{22} \\ F_{25} \end{bmatrix} = 0 \qquad \ldots(21)$$

where J is the Jacobian matrix of order 19. Thus a better solution is found by repeating the procedure till achieving the desired accuracy.

4. Results and Discussions

The numerical computations for both the methods in case of small Reynolds number R (= 0.01), have been carried out by setting $S = 1$ ($S = R^{\frac{1}{2}} S_1$) and $\tau_1 = -1$ ($\tau_1 = -R\tilde{\tau}_1$). The values of the velocity function f' in this case for both the methods at different nodal points have been given in Table I and represented graphically in Fig. 1. It is clear that the values of the velocity function f' obtained through

TABLE I

Values of the velocity function f' at $S = 1$ and $\tau_1 = -1$ for small Reynolds number R (= 0.01) by the perturbation (series solution) and numerical methods

ζ/f'	Series solution	Numerical solution
1.0	0.00000000	0.00000000
0.9	0.24998043	0.25263160
0.8	0.49472493	0.49528357
0.7	0.72521006	0.72239611
0.6	0.93463705	0.92898972
0.5	1.11800600	1.11074460
0.4	1.27174000	1.26405130
0.3	1.39336230	1.38603600
0.2	1.48122800	1.47456010
0.1	1.53431410	1.52819170
0.0	1.55206750	1.54614810
−0.1	1.53431410	1.52819170
−0.2	1.48122800	1.47456010
−0.3	1.39336230	1.38603600
−0.4	1.27174000	1.26405130
−0.5	1.11800600	1.11074460
−0.6	0.93463705	0.92898972
−0.7	0.72521006	0.72219611
−0.8	0.49472493	0.49528357
−0.9	0.24998043	0.25263160
−1.0	0.00000000	0.00000000

the perturbation and numerical methods are very much close to each other. The shape of the graph is parabolic with vertex lying on the line $\zeta = 0$ showing the flow through a channel in case of small Reynolds number (i.e for small suction) is almost the Couette flow.

The values of f' for $R = 9{,}25$ at $S = 5$ and $\tau_1 = -1$ for both the methods are given in Table II and represented graphically through Fig. 2 while the behaviour of f' with ζ for $S = 3, 6, 9, 12$ at $R = 9$, $\tau_1 = -1$ is shown in Fig. 3. It is seen from the Figs. 2 and 3 that the velocity function f' which is symmetric about the line $\zeta = 0.0$ attains its maxima at the middle of the gaplength and minima near the walls of the channel. It is also evident from these Figs. 2 and 3 that the back flows occur near the walls of the channel and an increase in R and S increases the value of f' in the middle of the gaplength while decreases near the walls of the channel.

Figure 4 depicts the velocity function f' with ζ for $\tau_1 = -1, -2, -4, -8$ at $R = 9$ and $S = 5$. It is seen that the velocity function f' decreases with an increase in absolute value of τ_1 in the gaplength while increases near the walls of the channel.

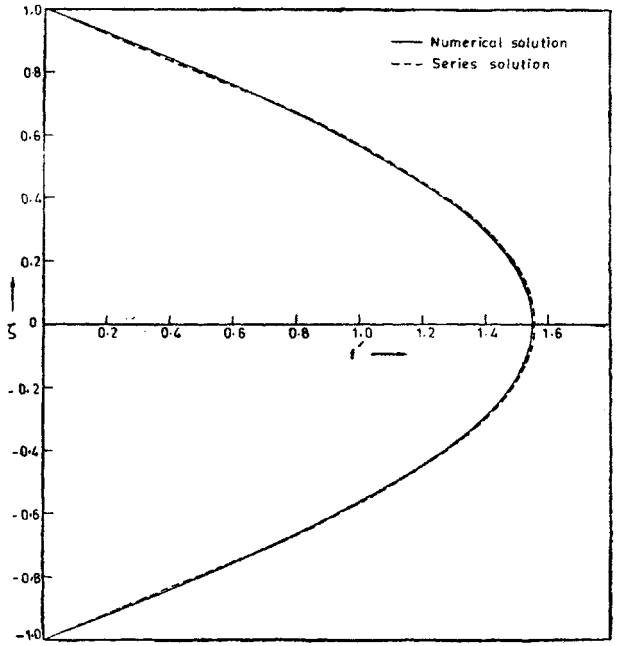


FIG. 1 Variation of the velocity function f' with ζ at $S = 1$ and $\tau_1 = -1$ for small Reynolds number R (= 0.01).

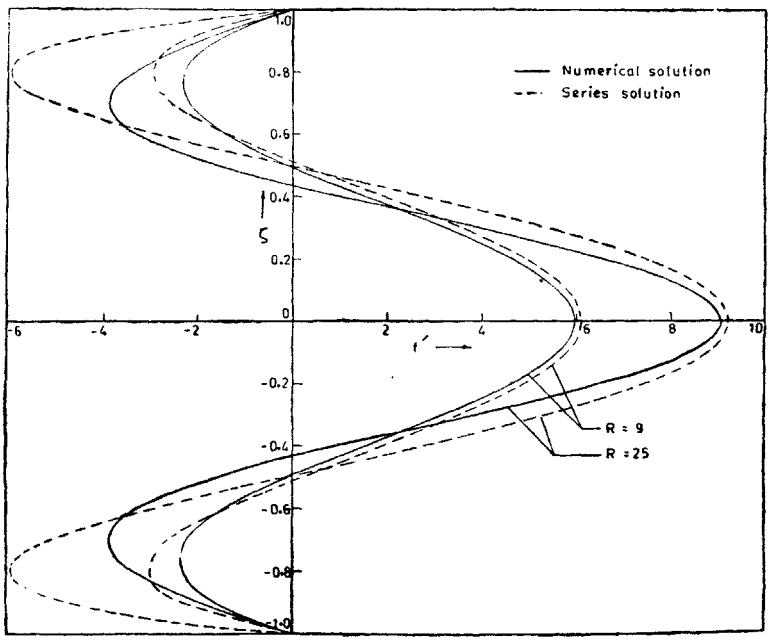


FIG. 2. Variation of the velocity function f' with ζ at $S = 5$ and $\tau_1 = -1$ for moderately large Reynolds number R (= 9,25).

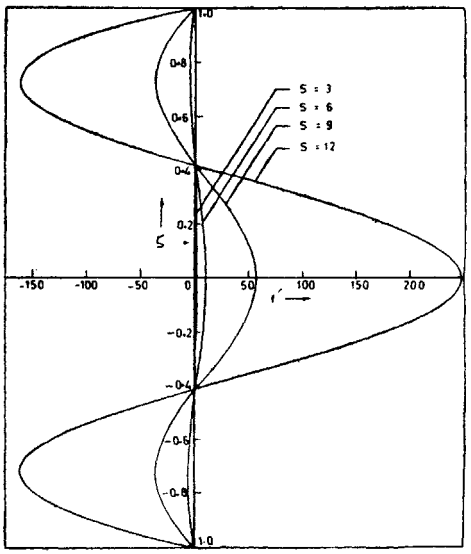


FIG. 3. Variation of the velocity function f' with ζ for different values of S at $R = 9$ and $\tau_1 = -1$.

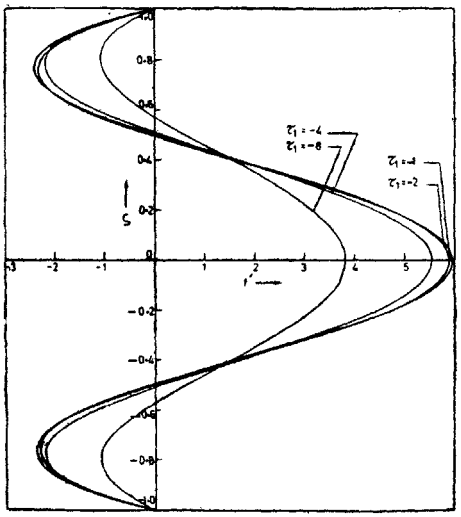


FIG. 4. Variation of the velocity function f' with ζ for different values of τ_1 at $R = 9$ and $S = 5$.

It is clear from Tables I and II and Figs. 1 and 2 that the deviation in the values obtained by perturbation and numerical methods increases with an increase in R. In perturbation method we have considered the terms upto $O(R^3)$ as it is very difficult and almost impractical to obtain a higher order perturbation solution. The results

TABLE II

Values of the velocity function f' at $S = 5$ and $\tau_1 = -1$ for moderately large Reynods number R by the perturbation and numerical methods

	R			
	9		25	
ζ/f'	Series solution	Numerical solution	Series solution	Numerical solution
1.0	0.00000000	0.00000000	0.00000000	0.00000000
0.9	−2.26986960	−1.55874650	−4.56851940	−1.87601450
0.8	−2.99701910	−2.28979410	−5.92539290	−3.31416500
0.7	−2.58757280	−2.21202950	−5.08616820	−3.86138070
0.6	−1.40945210	−1.42152830	−2.91209400	−3.27034830
0.5	0.20486765	0.09070165	−0.09898673	−1.55879230
0.4	1.96027390	1.54666990	2.81609910	0.99022314
0.3	3.60370940	3.22168890	5.43702010	3.88773420
0.2	4.93004550	4.66414980	7.48615590	6.54576950
0.1	5.78787010	5.63978880	8.78278970	8.40845350
0.0	6.08425640	5.98911160	9.22581920	9.07778500
−0.1	5.78787010	5.63978880	8.78278970	8.40845350
−0.2	4.93004550	4.66414980	7.48615590	6.54576960
−0.3	3.60370940	3.22168890	5.43702010	3.88773420
−0.4	1.96027390	1.54666990	2.81609910	0.99022314
−0.5	0.20486765	−0.09070165	−0.09896730	−1.55879230
−0.6	−1.40945210	−1.42152830	−2.91209400	−3.27034830
−0.7	−2.58757280	−2.21202950	−5.08616820	−3.86138070
−0.8	−2.99701910	−2.28979410	−5.92539290	−3.31416500
−0.9	−2.26986960	−1.55874650	−4.56851400	−1.87601460
−1.0	0.00000000	0.00000000	0.00000000	0.00000000

determined by third order perturbation method may be reliable only in case of small Reynolds number R and are very much close to the results obtained by numerical method for large Reynolds number.

The convergence rate of the Newton-Raphson method is fast. Only few iterations are needed to obtain a six digit accuracy. In this problem the solution converges only for moderately large values of R which indicates about the limitation of the problem.

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