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A NOTE ON COINCIDENCE THEOREMS AND KKM MAPS

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In this note the concepts of Q and Q^* applications are formulated. Using Lassonde's version of the KKM theorem some coincidence Theorems are unified and an existence theorem for a class of inequalities has also been obtained.

Browder[2] proved a fixed point theorem for a multi-map defined on a compact convex subset of a Hausdorff topological vector space and deduced a number of interesting results including *Ky* Fan's fixed point theorem[3]. Fan[4] obtained an infinite dimensional version of the KKM theorem from which Browder's theorem could be deduced. Tarafdar[7] extended Browder's fixed point theorem and using this extension he also obtained an existence theorem for a class of variational inequalities. Using the concepts of ϕ and ϕ^* applications Ben-El-Mechaiekh *et al.*[1] further extended these results of Browder and Tarafdar. In another direction Lassonde[5] generalized the KKM theorem in the setting of convex spaces more general than topological vector spaces and deduced some coincidence theorems extending Browder's theorem. Browder[2] obtained a coincidence theorem using *Ky* Fan's fixed point theorem. A recent result due to Sehgal *et al.*[6] to supplements Browder's coincidence theorem and indeed it leads to Browder's fixed point theorem. In this note we formulate the notions of Q and Q^* applications and using Lassonde's KKM theorem prove coincidence theorems unifying those of Ben-El-Mechaiekh *et al.*[1], Browder[2], Lassonde[5], Sehgal *et al.*[6] and Tarafdar[7]. An existence theorem for a class of inequalities has also been obtained.

We refer to Lassonde[5] for the following concepts of a convex space, a *c*-compact set, and a compactly closed (open) set and state below his generalization of the KKM theorem.

Definition 1.1—Lex X be a convex set in a vector space and $D \subset X$ an arbitrary subset. A multifunction

$$G\colon D \to 2^X \text{ is called KKM if } \operatorname{conv}\{x_1,\ldots,x_n\}$$

$$\subset \bigcup_{i=1}^{n} G(x_i) \text{ for each finite subset } \{x_1,\ldots,x_n\} \subset D.$$

Definition 1.2—A convex space is a convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets.

Definition 1.3 —Let X be a convex space. A nonempty set $K \subset X$ is called a c-compact set if for each finite sulset $\hbar \subset X$ there is a compact convex set $K_{\hbar} \subset X$ such that $K \cup \hbar \subset K_{\hbar}$. Let X be any quasi-complete convex set in a Hausdorff locally convex topological vector space (abbreviated as l.c.s). Every nonempty pre-compact set in X is c-compact.

Definition 1.4—Let Y be a topological space. A set $B \subset Y$ is said to be compactly closed (open respectively) in Y if for every compact set $L \subset Y$ the set $B \cap L$ is closed (open, respectively) in L.

For examples Lassonde[5] may be referred.

Theorem 1.1[5]—Let D be an arbitrary set in a convex space X, Y any topological space and F: $D \to 2^Y$ a multifunction having the following properties :

(i) for each $x \in D$, $F(x)$ is compactly closed in Y,

(ii) for some continuous map $s : X \to Y$, the multimap G: $D \to 2^Y$ given by $G(x) = s^{-1}(F(x))$ is KKM (i.e. for every finite subset $\{x_1, \ldots x_n\} \subset D$, conv $\{x_1 \ldots x_n\}$ $\subseteq \bigcup_{i=1}^{n} G(x_i)$

(iii) for some c-compact set $K \subset X$, $\cap \{F(x) : x \in K \cap D\}$ is compact. Then

$$\cap \{F(x) : x \in D\} \neq \phi.$$

2. Coincidence Theorems

First we formulate concepts of Q and Q^* applications.

Definition 2.1—Let X and Y be topological spaces, A: $X \to 2^Y$. A is a Q application if and only if (i) X is convex, (ii) $A^{-1}(Y)$ is convex for each $y \in Y$, (iii) there exists a selection $\hat{A}$ of A i.e. $\hat{A}$: $X \to 2^Y$ such that $\hat{A}(x) \subset A(x)$ for every $x \in X$ such that $\hat{A}$ is surjective and $\hat{A}(U)$ is compactly open in Y for each $U \subseteq X$.

A is a Q^* application if and only if (i) Y is convex, (ii) Ax is convex for each $x \in X$, (iii) there exists a selection $\hat{A} : X \to 2^Y$ such that $\hat{A}(x) \neq \phi$ for each $x \in X$ and $\hat{A}^{-1}(U)$ is compactly open in X for each $U \subset Y$.

The following coincidence theorem based on Theorem 1.1 is a generalization of a theorem of Lassonde's.

Theorem 2.1—Let X be a convex space and Y any topological space, $S : X \to 2^Y$ is a Q application. If there exists a c-compact set $K \subset X$ such that $Y / \bigcup_{x \in K} \hat{S}(x)$ is compact where $\hat{S}$ is the selection of S. Then for every continuous map $s: X \to Y$ there exists an x_0 with $sx_0 \in Sx_0$.

PROOF : We apply Theorem 1.1 to the function $F: X \to 2^Y$ defined by $Fx = Y/\hat{S}(x)$, $x \in X$. As $\hat{S}(x)$ is compactly open in Y, Fx is compactly closed in Y for each $x \in X$. Now, define $G: X \to 2^X$ by $G(x) = s^{-1} F(x)$. Since for the c-compact set K, $\bigcap_{x \in K} F(x)$ is compact if G were a KKM map by Theorem 1.1 it will follow that $\bigcap_{x \in K} F(x) \neq \phi$ contradicting the surjectivity of S. So G is not a KKM map. There exists $\{x_1, \ldots x_n\} \subseteq X$ and $x_0 \in \text{Conv}\{x_1 \ldots x_n\}$ such that $x_0 \notin \bigcup_{i=1}^{n} G(x_i)$ and so $sx_0 \notin \bigcup_{i=1}^{n} F(x_i)$ or equivalently $sx_0 \in \bigcup_{i=1}^{n} \hat{S}(x_i)$. This means $x_i \in \hat{S}^{-1}(sx_0) \subset S^{-1}(sx_0)$ for each $i = 1, 2 \ldots n$ and as $S^{-1}(sx_0)$ is convex $x_0 \in S^{-1}(sx_0)$ or $sx_0 \in Sx_0$.

Remark 2.1—Upon setting $\hat{S} = S$ the above theorem reduces to a coincidence Theorem 1.1 of Lassonde[5].

The following fixed point theorem generalizes that of Ben-El-Mechaick *et al.*[1] (Theorem 3.3) hence those of Browder[2] and Tarafdar[7].

Theorem 2.2—Let X be a convex space, $T : X \to 2^X$. Assume that

(i) T is a Q^* application and for some c-compact set $K \subset X$, $X/ \bigcup_{x \in K} T^{-1}(x)$ is compact

or

(ii) T is a Q application and for some c-compact set $K \subseteq X$, $X/ \bigcup_{x \in K} \hat{T}(x)$ is compact.

Then there exists $x_0 \in X$ such that $x_0 \in Tx_0$.

PROOF : Define $Sx = T^{-1}(x)\ (T(x))$ in case (i) and (ii) holds. It is readily verified that all the conditions of Theorem 2.1 are satisfied for $X = Y$ and $s =$ identity in both the cases (i) and (ii). Thus in either case there exists $x_0 \in Tx_0$.

The next theorem generalizes the coincidence theorem due to Sehgal *et al.*[6] in the setting of compact convex spaces.

Theorem 2.3—Let X and Y be convex spaces and $S, T : X \to 2^Y$ two multifunctions satisfying the following conditions :

(i) S is a Q application with $S^{-1}(Tx)$ convex for every $x \in X$,

(ii) there exists a selection $\hat{T} : X \to 2^Y$ of T with $\hat{T}(x) \neq \phi$ for each $x \in X$ and $\hat{T}^{-1}(U)$ is compactly open in X for each compactly open $U \subset Y$,

(iii) there exists a c-compact set $K \subset X$ with $X / \bigcup_{x \in K} \hat{T}^{-1}(\hat{S}(x))$ compact.

Then there exists $x_0 \in X$ with $Sx_0 \cap Tx_0 \neq \phi$.

PROOF: Suppose that $Tu \cap Su = \phi$ for every $u \in X$, then $u \notin T^{-1}(Su)$. Consequently $u \notin \hat{T}^{-1}(\hat{S}(u))$. Define $F : X \to 2^X$ by $Fu = X / \hat{T}^{-1}(\hat{S}(u))$. Since $\hat{S}(u)$ is compactly open in Y and $\hat{T}^{-1}(U)$ is compactly open in X for every compactly open U in Y, Fu is compactly closed in X and F is a KKM map. For $x_0 \in \text{conv}(x_1, \ldots x_n)$, $x_0 \notin \bigcup_{i=1}^{n} F(x_i)$ implies $x_0 \in \hat{T}^{-1}(\hat{S}(x_i)) \subset T^{-1}(Sx_i)$ and $x_i \in S^{-1}(Tx_0)$ for $1 \leqslant i \leqslant n$. As $S^{-1}(Tx_0)$ is convex $x_0 \in S^{-1}(Tx_0)$ i.e. $Sx_0 \cap Tx_0 \neq \phi$ a contradiction to our assumption. By (iii) for the c-compact set $K \subset X$, $\bigcap_{x \in K} F(x)$ is compact. Thus all the conditions of Theorem 1.1 are satisfied with s = identity on X. So there exists $x_0 \in X$ with $x_0 \in \bigcap_{u \in X} Fu$ or $x_0 \notin \hat{T}^{-1}(\hat{S}(u))$ for any $u \in X$. As $\hat{T}(x_0) \neq \phi$ and $\hat{S}$ is surjective we have arrived at a contradiction. Thus there exists $x_0 \in X$ with $Sx_0 \cap Tx_0 \neq \phi$.

The following concept of a function subordinate to a projectionally concave function leads to a generalization of an existence theorem due to Lassonde[5] (Proposition 1.4).

Definition 2.2—Let X be a convex space. A map $g : X \times X \to R \cup \{+\infty\}$ is said to be subordinate to a projectionally concave function if there exists $f : X \times X \to R$ $U \{+\infty\}$ such that $g(x, y) \leqslant f(x, y)$ for every $(x, y) \in X \times X$, $f(x, x) \leqslant 0$ and for every $y \in X$, $x \to f(x, y)$ is concave on X.

For example the function $g(x, y) = \frac{-\chi_Q(x)}{2} Y$ is subordinate to the projectionally concave function $f(x, y) = -\chi_Q(x) Y^2$ where $X = [0, 1]$ and χ_Q is the characteristic function of the rationals in $[0, 1]$.

Theorem 2.4—Let X be a convex space and $\phi : X \to R \cup \{+\infty\}$ a convex lower semicontinuous function. Suppose $g : X \times X \to R\ U\ \{+\infty\}$ is subordinate to a projectionally concave function and that there exists a c-compact set $K \subseteq X$ and a compact set $L \subseteq X$ such that for every $y \in X/L$ there exists $x \in K$ such that $g(x, y) + \phi(y) > \phi(x)$ (coercivity condition). Suppose for $x \in X$, $y \to g(x, y)$ is lower semi-continuous on compact subsets of X, then there exists $y_0 \in X$ such that $g(x, y_0) + \phi(y_0) \leqslant \phi(x)$ for every $x \in X$.

PROOF: Define $Tx = \{y \in X : f(x, y) + \phi(y) > \phi(x)\}$ and $\hat{T}(x) = \{y \in X;$ $g : (x, y) + \phi(y) > \phi(x)\}$ where g is subordinate to f. Since $g(x, y) \leqslant f(x, y)$

$\hat{T}(x) \subseteq T(x)$ for every $x \in X$, by the concavity of the function $x \to f(x, y)$ $T^{-1}(y)$ is convex for every $y \in X$. As $y \to g(x, y)$ is lower semicontinuous on compact subsets of X $T(X)$ is compactly open for each $x \in X$. From the coercivity condition it can be verified that for the c-compact set K, $X / \bigcup_{x \in K} \hat{T}(x)$ is compact. If T were surjective by Theorem 2.2 there would be an $x_0 \in X$ with $x_0 \in T(x_0)$ contradicting that $f(x_0, x_0) \leqslant 0$. So T is not surjective and there exists $y_0 \in X$ such that $y_0 \notin \hat{S}(x)$ for any $x \in X$. Thus $g(x, y_0) + \varphi(y_0) \leqslant \varphi(x)$ for every $x \in X$.

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