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ON THE REDUCTION OF DIRICHLET-NEWTON PROBLEMS TO THE WING EQUATION

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This paper presents a special method for solving problems with "Dirichlet-Newton" boundary conditions. The proposed method consists of reducing the problem to a discrete problem by means of the finite Fourier transform as in Cherski[1]. This in turn is transformed to the aircraft wings singular integro-differential equation. Using the orthogonal Chebyshev polynomials, the latter equation can be reduced to an infinite system of algebraic equations. We illustrate the method by using a typical problem: the stationary heat equation within the unit circle. However, its application to initial mixed problems of the "Dirichlet-Newton" type may constitute the subject of a consequential paper.

1. FORMULATION OF A TYPICAL PROBLEM AND ITS REDUCTION TO A DISCRETE PROBLEM

We consider the mixed boundary value problem

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0, \; (r < 1, \; |\theta| < \pi) \qquad \text{...(1.1)}$$

$$T(1, \theta) = 0 \; |\theta + \tfrac{1}{2}\pi| \leqslant \alpha \qquad \text{...(1.2)}$$

$$\frac{\partial T(1, \theta)}{\partial r} = h(T - f) \; |\theta + \tfrac{1}{2}\pi| > \alpha \qquad \text{...(1.3)}$$

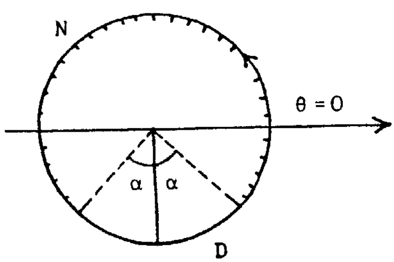

FIG. 1.

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where f is a given function of θ. The boundary conditions can also be written in the form

$$T(1,\theta) = \varphi_-(\theta) = \begin{cases} 0 & |\theta + \tfrac{1}{2}\pi| \leqslant \alpha \\ \text{underemined} & |\theta + \tfrac{1}{2}\pi| > \alpha \end{cases} \qquad \text{...(1.4)}$$

$$\frac{\partial T(1,\theta)}{\partial r} - h(T - f) = \varphi_+(\theta) = \begin{cases} \text{undetermined} & |\theta + \tfrac{1}{2}\pi| \leqslant \alpha \\ 0 & |\theta + \tfrac{1}{2}\pi| > \alpha \end{cases} \qquad \text{...(1.5)}$$

Like the methods presented earlier[1,3,4], the key idea of this paper is based on the determination of one of the two functions $\varphi(\theta)_\pm$ in such a way that the two conditions (1.4) and (1.5) become compatable. On applying the finite Fourier transform with respect to θ

$$G_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta)\, e^{-in\theta}\, d\theta, \quad g(\theta) = \sum_{n=-\infty}^{\infty} G_n\, e^{in\theta} \qquad \text{...(1.6)}$$

to eqn. (1.1), solving it, and then applying it to (1.4) and (1.5) we obtain

$$T_n(r) = A_n\, r^{|n|} \qquad \text{...(1.7)}$$

$$T_n(1) = \Phi_{n-} \qquad \text{...(1.8)}$$

$$\frac{\partial T_n(1)}{\partial r} = h\,[T_n(1) - F_n] + \Phi_{n+}. \qquad \text{...(1.9)}$$

Substituting (1.7) into (1.8) and (1.9), and eliminating A_n, we get the discrete problem

$$|n|\, \Phi_{n-} = h\,\Phi_{n-} - h\,F_n + \Phi_{n+}. \qquad \text{...(1.10)}$$

2. The Transformation of the Discrete Problem to the Wing Equation

On performing the inverse Fourier transform to (1.10) we get

$$\frac{1}{\pi i}\,\frac{d}{d\theta} \int_{-\pi}^{\pi} \frac{\varphi_-(y)\,dy}{1 - e^{i(\theta - y)}} = h\,\varphi_-(\theta) - h\,f(\theta) + \varphi_+(\theta) \qquad \text{...(2.1)}$$

where the integral in the left-hand side is understood in the sense of the Cauchy principal value. Recalling that $\varphi_+(\theta) = 0$ for $|\theta + \tfrac{1}{2}\pi| > \alpha$, equation (2.1) becomes

$$\frac{1}{\pi i}\,\frac{d}{d\theta} \int_{|y + \frac{1}{2}\pi| > \alpha} \frac{\varphi_-(y)\,dy}{1 - e^{i(\theta - y)}} = h\,\varphi_-(\theta) - h\,f(\theta)$$

then making use of the assumption that $\varphi_-(y)$ vanishes at the two end points $|y + \tfrac{1}{2}\pi| = \alpha$

[see eqn (1.4)], we obtain

$$\frac{1}{\pi i} \int_{|y+\frac{1}{2}\pi| \geqslant \alpha} \frac{\varphi_-'(y)\,dy}{1-e^{i(\theta-y)}} = h\,\varphi_-(\theta) - h\,f(\theta).$$

Setting

$$e^{iy} = \tau,\ e^{i\theta} = \tau_0 \qquad \text{...(2.2)}$$

and let L be the arc of the unit circle in the complex τ-plane cut in the positive sense and joining the two points $-ie^{i\alpha}$, respectively. Thus, the above equation takes the form

$$h\,\tilde{\varphi}_-(\tau_0) - \frac{1}{\pi i}\int_L \frac{\tau\,\tilde{\varphi}_-'(\tau)\,d\tau}{\tau-\tau_0} - h\,\tilde{f}(\tau_0) = 0.$$

This equation can be simplified by using the equality

$$\int_L \frac{\tau\varphi'(\tau)\,d\tau}{\tau-\tau_0} = \int_L \frac{\tau-\tau_0}{\tau-\tau_0}\varphi_-'(\tau)\,d\tau + \tau_0\int_L \frac{\varphi'(\tau)}{\tau-\tau_0}\,d\tau$$

$$= \tau_0 \int_L \frac{\varphi_-'(\tau)}{\tau-\tau_0}\,d\tau.$$

Thereby we have

$$h\,\tilde{\varphi}_-(\tau_0) - \frac{\tau_0}{\pi i}\int_L \frac{\tilde{\varphi}_-'(\tau)\,d\tau}{\tau-\tau_0} - h\,\tilde{f}(\tau_0) = 0. \qquad \text{...(2.3)}$$

Now, replacing in eqn. (2.3) the variable τ by x, where

$$\tau = -i\,\frac{x-i}{x+i} \qquad \text{...(2.4)}$$

it turms out that x varies along the real interval $\left[-\cot\frac{\alpha}{2}, \cot\frac{\alpha}{2}\right]$ as τ covers L, and we get

$$h\,\bar{\varphi}_-(x_0) + \frac{x_0-i}{2\pi}\int_{-\cot\frac{\alpha}{2}}^{\cot\frac{\alpha}{2}} \frac{(x+i)}{x-x_0}\,\bar{\varphi}_-'(x)\,dx - h\bar{f}(x_0) = 0.$$

Again $\bar{\varphi}_-\left(\pm\cot\frac{\alpha}{2}\right) = 0$, and we arrive at the integro-differential equations of the theory of aircraft wings of finite span :

$$h\ \bar{\varphi}_-(x_0) + \frac{1+x_0^2}{2\pi}\int_{-\cot\frac{\alpha}{2}}^{\cot\frac{\alpha}{2}} \frac{\bar{\varphi}'_-(x)}{x-x_0}\,dx - h\ \bar{f}(x_0) = 0. \qquad \ldots(2.5)$$

It can be written in the standard form

$$\frac{\varphi^*_-(t_0)}{\zeta^2+t_0^2} + \frac{1}{\pi\lambda}\int_{-1}^{1}\frac{\varphi^{*\prime}_-(t)}{t-t_0}\,dt - \frac{f^*(t_0)}{\zeta^2+t_0^2} = 0 \qquad \ldots(2.6)$$

where

$$x = t\cot\frac{\alpha}{2},\ x_0 = t_0\cot\frac{\alpha}{2} \qquad \ldots(2.7)$$

and

$$\zeta = \tan\frac{\alpha}{2},\ \lambda = 2h\,\zeta. \qquad \ldots(2.8)$$

3. The Transformation of the Wing Equation to the Algebraic System

Because of its great interest, many studies for the solution of the wing equation (2.5) have been undertaken. The reader is referred, for example, to Muskhelishvili[5] (p. 373), Multhopp[6] and the references given in both. However, the method suggested by Morar and Popov[2] is probably the most suitable one for purpose. It is appropriate to apply their results providing a brief arrangement of the procedures neccessary for the solution of the important class $\psi(t_0) = \zeta^2 + t_0^2$ [i. e. eqn. (2.6)] and many Dirichlet-Newton problems thereby. This statement will be clarified in section 5. In addition, the last formula in their influential work which is fundamental for the solution of equations of the class (2.6) and any other more general classes than those considered there is incorrect. The correction is given here [eqn. 3.10)].

The solution of equation (2.6) can be thought of in the form

$$\varphi^*_-(t_0) = \sqrt{1-t_0^2}\ \sum_{m=0}^{\infty} Y_m\,U_m(t_0) \qquad \ldots(3.1)$$

where $U_m(t_0)$ are Chebyshev polynomials o the second kind. The choice (3.1) is based on the conditions

$$\varphi^*_-(1) = \varphi^*_-(-1) = 0. \qquad \ldots(3.2)$$

Thus, we have

$$\frac{1}{\pi\lambda}\int_{-1}^{1}\frac{\varphi_{-}^{*\prime}(t)}{t-t_0}\,dt=\frac{-1}{\pi\lambda}\sum_{m=0}^{\infty}(m+1)\,Y_m\int_{-1}^{1}\frac{T_{m+1}(t)}{\sqrt{1-t^2}\,(t-t_0)}\,dt$$

$$=\frac{-1}{\lambda}\sum_{m=0}^{\infty}(m+1)\,Y_m\,U_m(t_0). \qquad \ldots(3.3)$$

Here, use has been made of the result

$$\int_{-1}^{1}\frac{T_m(t)\,dt}{(t-t_0)\sqrt{1-t^2}}=\begin{cases}0 & (m=0)\\ \pi U_{m-1}(t_0) & (m=1,2,3,\ldots)\end{cases} \qquad \ldots(3.4)$$

where $T_m(t)$ are Chebyshev polynomials of the first kind. Substituting (3.1) and (3.3) into eqn. (2.6), multiplying by $\sqrt{1-t_0^2}\;U_k(t_0)$, and integrating over the interval $(-1, 1)$, we obtain

$$-\frac{\pi}{2\lambda}(k+1)\,Y_k+\sum_{m=0}^{\infty}Y_m\,C_{m,k}=b_k \qquad \ldots(3.5)$$

where

$$C_{m,k}=\int_{-1}^{1}\frac{\left(1-t_0^2\right)}{\zeta^2+t_0^2}\,U_m(t_0)\,U_k(t_0)\,dt_0 \qquad \ldots(3.6)$$

and

$$b_k=\int_{-1}^{1}\frac{f(t_0)}{\zeta^2+t_0^2}\sqrt{1-t_0^2}\;U_k(t_0)\,dt_0. \qquad \ldots(3.7)$$

From eqns. (3.1) and (3.3) it can easily be seen that the first two terms in eqn. (2.6) will be converted to a combination of just the Chebyshev polynomials that should take place in the expansion of $\varphi^*_-(t_0)$ to within a multiplicative factor even in t_0. In other words, if $\varphi^*_-(t)$ is an even (odd) function, so will be the singular integral appearing in (2.6). Therefore, if $f^*(t_0)$ is even (odd), $\varphi^*_-(t_0)$ will also be even (odd). In this case, the algebraic system determining $\varphi^*_-(t_0)$ can be obtained by applying the replacements

$$k \to 2k, \; m \to 2m$$

$$(k \to 2k+1, \; m \to 2m+1) \qquad \ldots(3.8)$$

to the system (3.5) while the summation is retained over the same values $m = 0, 1, 2, \ldots$. In the general case, it is more convenient to break down the problem to an even and an odd problem.

Now, if

$$\frac{1}{\psi(t_0)} = \sum_{n=0}^{\infty} a_n \, U_n(t_0) \qquad \ldots(3.9)$$

then, converting to the variable θ where $t_0 = \cos\theta$, it can be shown that

$$\int_{-1}^{1} \frac{\left(1 - t_0^2\right)}{\psi(t_0)} U_m(t_0) \; U_k(t_0) \, dt_0$$

$$= -4(m+1) \sum_{n=0}^{\infty} \frac{a_n (k+1)(n+1) \cos^2\left[(n+m+k)\frac{\pi}{2}\right]}{[(m+1)^2 - (n-k)^2]\,[(m+1)^2 - (n+k+2)^2]}$$

$$(m, k = 0, 1, \ldots) \qquad \ldots(3.10)$$

but since

$$\frac{1}{\zeta^2 + t_0^2} = \frac{2}{\zeta} \sum_{n=0}^{\infty} (-)^n \left(\sqrt{1+\zeta^2} - \zeta\right)^{2n+1} U_{2n}(t_0), \; \zeta^2 > 0 \qquad \ldots(3.11)$$

as it is easy to verify by replacing this function on the left-hand side by its intergal representation (formula 3.893 (2) of Multhopp[6], reversing the order of integration, using the substitution $t_0 = \cos\theta$, and applying the formula 3.715 (18) of Gradshtein and Ryzhik[7], it follows from (3.10) that

$$C_{m,k} = \frac{8}{\zeta} \sum_{n=0}^{\infty} \frac{(-)^{n+1}(2n+1)(k+1)(m+1)\left(\sqrt{1+\zeta^2}-\zeta\right)^{2n+1} \cos^2\left[(2n+m+k)\frac{\pi}{2}\right]}{[(m+1)^2 - (2n-k)^2]\,[(m+1)^2 - (2n+k+2)^2]} \; . \qquad \ldots(3.12)$$

In contrast to the last formula of Morar and Popov[2], expression (3.12) is symmetric in m and k as might be expected [Definition 3.6)]. It is also clear that the coefficients $C_{m,k}$ assume nonzero values only when m and k are both even and odd together in accordance with the discussion below definition (3.7).

We note that for $\zeta^2 > 0$, $\sqrt{1+\zeta^2} - \zeta$ is always less than unity. It follows that the system (3.5) is quasiregular for $\zeta^2 > 0$ as can simply be verified :

$$\beta_k = \frac{2\lambda}{\pi(2k+1)} \sum_{m=0}^{\infty} [C_{2m,\ 2k}] \rightarrow 0 \text{ as } k \rightarrow 0$$

$$\beta'_k = \frac{\lambda}{\pi(k+1)} \sum_{m=0}^{\infty} [C_{2m+1,\ 2k+1}] \rightarrow 0 \text{ as } k \rightarrow 0. \qquad ...(3.13)$$

The case $\zeta = 0$ corresponds to the regular Dirichlet problem which can be solved without making use of the method presented in this paper.

4. Simple Application

A simplified example is considered here to give some idea about the effectiveness in truncating the system (3.5). We shall suppose that the temperature of the medium surrounding the arc $\left|\theta + \frac{\pi}{2}\right| > \alpha$ is constant. Thus we can take

$$f(\theta) = 1. \qquad ...(4.1)$$

A different value for that constant temperature will change all the coefficients b_k in the system (3.5) (and the solution of the problem thereby) through a multiplicative factor equal to this value. In this case, we have

$$b_{2k+1} = 0$$

$$b_{2k} = \int_{-1}^{1} \frac{1}{\zeta^2 + t_0^2} \sqrt{1 - t_0^2}\, U_{2k}(t_0)\, dt_0$$

$$= \int_{-1}^{1} \left[\frac{2}{\zeta} \sum_{m=0}^{\infty} (-)^m \left(\sqrt{1+\zeta^2} - \zeta\right)^{2m+1} U_{2m}(t_0) \right] U_{2k}(t_0) \sqrt{1 - t_0^2}\, dt_0$$

$$= (-)^k \frac{\pi}{\zeta} \left(\sqrt{1+\zeta^2} - \zeta\right)^{2k+1} \qquad ...(4.2)$$

and the problem is even.

Also, replacing hr by r in the condition (1.3), we note that the conditions (1.1) and (1.2) will not be influenced if the scale is suitably changed. Thus we can set

$$h = 1 \qquad ...(4.3)$$

without loss of generalization.

It is convenient to consider $\alpha = \frac{\pi}{2}$ since this case differs as far as possible from the usual uniform problems : $\alpha = \pi$ (Dirichlet) and $\alpha = 0$ (Newton). Denoting the solution truncated at the j-order by $Y_{2m}^{(j)}$, the following table provides an idea about the convenience of the truncation :

m	0	1	2	3	4
$Y_{2m}^{(3)}$	3.007687	−0.604036	0.093086	–	–
$Y_{2m}^{(4)}$	3.006269	−0.604441	0.095166	−0.018250	–
$Y_{2m}^{(5)}$	3.006995	−0.604471	0.095205	−0.018431	0.002365

In terms of its Fourier series, the function $\varphi_-(\theta)$ defined in (1.4) can clearly be thought of in the form

$$\varphi_-(\theta) = \sum_{n=1}^{\infty} C_n \sin n\theta$$

but from the symmetry we have

$$\varphi_-\left(\frac{\pi}{2} + \theta\right) = \varphi_-\left(\frac{\pi}{2} - \theta\right)$$

which yields

$$2 \sum_{n=1}^{\infty} C_{2n} (\sin 2n + 1)\, \theta = 0. \qquad \ldots(4.4)$$

Now, replacing in (4. 4) the coefficients C_{2n+1} by Y_{2n}, we arrive at exactly the same formula obtained from (3.1) by making use of the substitution

$$t_0 = x_0 = \cos\theta. \qquad \ldots(4.5)$$

Thus, from the uniqueness of the Fourier representation, we come to the conclusion that unknown function $\varphi_-(\theta)$ is found to be

$$\varphi_-(\theta) \approx 3.00699 \sin\theta - 0.60447 \sin 3\theta + 0.09521 \sin 5\theta$$
$$- 0.01843 \sin 7\theta + 0.00237 \sin 9\theta + \ldots \qquad \ldots(4.6)$$

5. Other Problems

In this section, other problems reduceable to equations of the class (2.6) are outlined.

The problem described by the eqns. (1.1) – (1.3) together with

$$T(\delta, \theta) = 0 \qquad 0 < \delta < 1. \qquad \ldots(5.1)$$

can be reduced to the discrete problem

$$|n| \; \Phi_{n-} - G_{|n|}\,(\delta) = h\,[\Phi_{n-} - F_{n-}] + \Phi_{n+} \qquad \text{...(5.2)}$$

where

$$G_{|n|}(\theta) = \begin{cases} [\ln \delta]^{-1} & n = 0 \\ \dfrac{2\,|n|\,\delta^{2|n|}}{\delta^{2|n|} - 1} & n \neq 0. \end{cases} \qquad \text{.. (5.3)}$$

The steps of the reduction run on the same pattern as before. The problem (5.2) differs from (1.10) only by the term involving $G_{|n|}\,(\delta)$ which vanishes rapidly. It is then a simple matter to see that eqn. (5.2) can be transformed to an equation of the class (2.6).

Further, if (5.1) is replaced by the mixed condition

$$T(\delta, \theta) = 0 \qquad \left|\theta + \frac{\pi}{2}\right| \leqslant \beta$$

$$\frac{\partial T(\delta, \theta)}{\partial r} = h\,(T - v) \qquad \left|\theta + \frac{\pi}{2}\right| > \alpha \qquad \text{...(5.4)}$$

where v is a given function of θ, the new problem is reduceable to

$$\left.\begin{aligned} |n|\,\Phi_{n-} - G_{|n|}\,(\delta)\,\Phi_{n-} + Q_{|n|}(\delta)\,\Psi_{n-} &= h\,(\Phi_{n-} - F_n) + \Phi_{n+} \\ |n|\,\Psi_{n-} - G_{|n|}(\delta)\,\Psi_{-n} + Q_{|n|}(\delta)\,\Phi_{n-} &= h\,\delta\,(\Psi_{n-} - V_n) - \delta\Psi_{n+} \end{aligned}\right\}. \qquad \text{...(5.5)}$$

where Ψ_{n-} are the Fourier coefficients of the undetermined temperature at the internal boundary, and

$$Q_{|n|}(\delta) = \delta^{-|n|}\,G_{|n|}\,(\delta) \quad \forall\, n. \qquad \text{...(5.6)}$$

In this case we will have two equations of the class (2.6), and there solution can be founded as before.

An example for a square region represented in the following figure

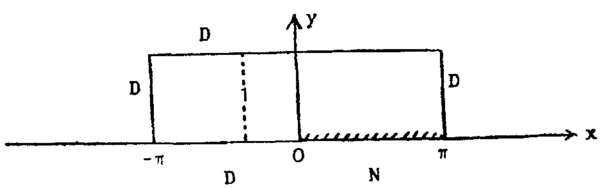

FIG. 2.

A Dirichlet condition is indicated by D, and N stands for Newton. The corresponding discrete problem is

$$| n | \Phi_{n-} + R_{|n|} \Phi_{n-} = h [\Phi_{n-} - F_n] - \Phi_{n+} \qquad \ldots(5.7)$$

where

$$R_{|n|} = \begin{cases} 1 & n = 0 \\ \dfrac{|n|}{\tanh |n|} - |n| & n \neq 0. \end{cases} \qquad \ldots(5.8)$$

Problems of symmetry properties other than all that considered so far can be solved by a slight modification. As an example for problems symmetric around $\theta = 0$, condition (2.4) should be replaced by

$$\tau = \frac{x - i}{x + i}. \qquad \ldots(5.9)$$

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Errata

ON GENERALIZED STIEFEL MANIFOLDS

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