Indian J. pure appl. Math., 17 (11): 1270-1275, November 1986

FIXED POINT THEOREMS FOR KANNAN MAPPINGS

ISMAT BEG AND AKBAR AZAM

Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan

(*Received* 27 *January* 1986)

The aim of the present paper is to prove the existence of fixed points for Kannan mappings in convex metric spaces. Banach spaces also fall in this category and as such our results generalize those of Kannan.

Takahashi[3] introduced the notion of convexity in metric spaces and generalized some fixed point theorems in Banach spaces. Subsequently Guay *et al.*, Talman[4] among others, have studied fixed point theorems in convex metric spaces. In this paper we prove existence of fixed points for Kannan mappings in convex metric spaces.

1. PRELIMINARIES

Let X be a metric space. A mapping $W: X \times X \times [0, 1] \to X$ is said to be a convex structure on X, if for all x, y in X and $\lambda \in [0, 1]$, the following condition is satisfied

$$d(u, W(x, y, \lambda)) \leqslant \lambda\, d(u, x) + (1 - \lambda)\, d(u, y), \text{ for all } u \in X. \qquad \ldots (1)$$

A metric space with convex structure is called convex metric space. A subset K of a convex metric space X is said to be convex if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$. A convex metric space X is said to have property (C) if every decreasing net of nonempty closed and convex subsets of X has nonempty intersection.

Remarks: (i) Every Banach space and each of its convex subset is a conxex metric space.

(ii) Every reflexive Banach space has property (C).

(iii) Every weakly compact convex subset of a Banach space, has property (C).

(iv) There are many convex metric spaces which can not be imbedded in any Banach space.

For details we refer to Takahashi[3] and Guay *et al.*[1].

Let K be a nonempty subset of a convex metric space X. A mapping $T: K \to K$ is said to be Kannan mapping if

$$d(Tx, Ty) \leqslant \frac{d(x, Tx) + d(y, Ty)}{2} \qquad \ldots (2)$$

for all $x, y \in K$.

Let T be a self mapping of a bounded subset K of a convex metric space X. Then T is said to have property (B) on K, if for every closed and convex subset F of K, which has non zero diameter and is invariant under T, there exists some $x \in F$ such that

$$d(x, Tx <) \sup_{y \in F} d(y, Ty).$$

2. Main Results

Theorem—1 Let T be a Kannan mapping of a nonempty bounded closed and convex subset K of a convex metric space X having property (C) into itself. If $\sup_{y \in F} d(y, Ty) < \delta(F)$, $\{\delta(F)$ being the diameter of $F)$ for every nonempty bounded closed and convex subset F of K which has non zero diameter and is mapped into itself by T, then T has a unique fixed point in K.

Proof : Let Γ be the family of all bounded closed and convex subsets of K, mapped into itself by T. Obviously Γ is nonempty and has a minimal element S, S being minimal with respect to being nonempty bounded closed and convex and invariant under T. If $\delta(S) = 0$, then the point in S is a fixed point of T. Suppose $\delta(S) > 0$. For any $x, y \in S$, we have

$$d(Tx, Ty) \leqslant \frac{d(x, Tx)}{2} + \frac{d(y, Ty)}{2} \leqslant \sup_{y \in S} d(y, Ty).$$

Hence $T(S)$ is contained in the closed sphere S_0 with Tx as centre and $\sup_{y \in S} d(y, Ty)$ as radius. Also $S \cap S_0$ is invariant under T. Therefore by the minimality of S it follows that $S \subset S_0$.

Hence for any arbitrary but fixed $x \in S$, we have

$$\sup_{y \in S} d(Tx, y) \leqslant \sup_{y \in S} d(y, Ty). \qquad \ldots(3)$$

Let

$$S' = [z \in S: \sup_{y \in S} d(z, y) \leqslant \sup_{y \in S} d(y, Ty)\}.$$

Obviously S' is nonempty.

For any $u, v \in S'$ and $\lambda \in [0, 1]$, we have

$$d\{W(u, v, \lambda), y\} \leqslant \lambda d(u, y) + (1 - \lambda) d(v, y),$$

$$\leqslant \sup_{y \in S} d(y, Ty).$$

It follows that $W(u, v, \lambda) \in S'$ for all $u, v \in S'$ and $\lambda \in [0, 1]$. Therefore S' is convex.

Next, suppose that $z \in \mathrm{Cl}(S')$, closure of S'. Then there exists a sequence (z_n) in S' such that $z_n \to z$, and

$$d(z_n, y) \leqslant \sup_{y \in S} d(z_n, y) \leqslant \sup_{y \in S} d(y, Ty)$$

for all $y \in S$. Letting n tend to infinity, we have

$$d(z, y) \leqslant \sup_{y \in S} d(y, Ty).$$

It follows that $z \in S'$, and therefore S' is closed.

For all $z \in S'$, eqn. (3) implies that

$$\sup_{y \in S} (Tz, y) \leqslant \sup_{y \in S} d(Ty, y).$$

Using definition of S', we have $Tz \in S'$ for all $z \in S'$. Thus S' is invariant under T. Also $\delta(S') \leqslant \sup_{y \in S} d(y, Ty) < \delta(S)$, by hypothesis. Hence S' is a proper closed and convex subset of S, which contradicts the minimality of S.

Uniqueness—Suppose that T have two fixed points x and y. Then

$$d(x, y) = d(Tx, Ty) \leqslant \frac{d(x, Tx) + d(y, Ty)}{2} = 0.$$

It follows that $x = y$.

Theorem 2—Let X be a convex metric space having property (C) and K be a nonempty bounded closed and convex subset of X. Let $T : K \to K$ be a continuous Kannan mapping. Suppose T has property (B) over K. Then T has a unique fixed point in K.

PROOF : As in the previous theorem, let S be the minimalel ement in Γ with respect to being nonempty bounded closed and convex and invariant under T. If $\delta(S) = 0$, the theorem is obvious. If $\delta(S) \neq 0$, by property (B), there exists $x \in S$ such that

$$d(x, Tx) = r < \sup_{y \in S} d(y, Ty). \qquad \ldots(4)$$

Let $P = \{x \in S : d(x, Tx) \leqslant r\}$. If $x \in P$, then since

$$d(Tx, T^2 x) \leqslant \frac{d(x, Tx) + d(Tx, T^2 x)}{2}.$$

We have $d(Tx, T^2 x) \leqslant r$ which implies $Tx \in P$ for all $x \in P$. Hence it follows that $T(P) \subset P$.

Let $P' = \mathrm{ClCo}(TP)$, the closed and convex hull of TP. If $z \in P'$, then any one of the following three cases may arise :

(i) $z \in TP$ and since $TP \subset P$, hence $Tz \in TP \subset P'$.

(ii) $z \in Co\,(TP) = \bigcup_{i \in N} A_i$,

where

$$A_1 = W(TP \times TP \times [0, 1]),$$

$$A_2 = W(A_1 \times A_1 \times [0, 1]),$$

....

....

....

It follows that there exists some $m \in N$ such that $z \in A_m$. Applying principle of mathematical induction, we get

$$d(z, Ty) \leqslant \frac{r}{2} + \frac{d(Ty, y)}{2}$$

for all $z \in A_m$ and $y \in K$. Thus, $d(z, Tz) \leqslant r$, which implies $z \in P$ and hence $Tz \in TP \subset P'$.

(iii) z is a limit point of $Co\,(TP)$, then there exists a sequence (z_n) in $Co\,(TP)$ such that $z_n \to z$. Since T is continuous, we have $Tz_n \to Tz$ and

$$d(z, Tz) = \lim_{n \to \infty} d(z_n, Tz_n) \leqslant r.$$

It follows that $z \in P$ and $Tz \in TP \subset P'$.

Thus P' is closed and convex subset of S which is invariant under T and, for every element z of P', $d(z, Tz) \leqslant r$, which implies by equation (4), that P' is a proper subset of S. This contradicts the minimality of S. Hence $\delta(S) = 0$, and the point in S is a fixed point of T.

Theorem 3—Let X be a convex metric space having property (C) and H be a closed and convex subset of X. Let K be a nonempty bounded closed and convex subset of H. Let $T : K \to H$ be a continuous Kannan map such that

(i) T maps $\partial_H K$ the boundary of K relative to H, into K.

(ii) If F is any closed and convex subset of K which has non zero diameter and if G is a subset of F such that $TG \subset F$, then there exists $x \in G$ such that

$$d(x, Tx) \leqslant \sup_{y \in F} d(y, Ty).$$

Then T has a unique fixed point in K.

PROOF : Let Γ be the family of all closed and convex subsets E of H such that $E \cap K \neq \phi$ and $T : E \cap K \to E$. Obviously $H \in \Gamma$. Let $\{F_\alpha\}$ be a descending chain of subsets of Γ. Property (C) implies that $F \cap K \neq \phi$ where $F = \cap F_\alpha$. Because $T: F_\alpha \cap K \to F_\alpha$ for each α, thus $T: F \cap K \to F$. Hence by Zorn's lemma there exists a minimal element S in Γ, S being minimal with respect to being nonempty closed and convex and such that $S \cap K \neq \phi$ and $T: S \cap K \to S$.

If $\partial_S K = \phi$, then $S \subset K$ and $T: S \cap K \to S$ implies T maps S into S. Condition (ii) implies that T has property (B). Now Theorem 2 further implies that T has unique fixed point in S.

If $\partial_S K \neq \phi$, then $\partial_S K \subset \partial_H K$ and condition (i) implies that T maps $\partial_S K$ into K. Also T maps $S \cap K$ into S. Hence T maps $\partial_S K$ into $S \cap K$. If $\partial(S \cap K) = 0$, then $S \cap K$ contains only one element z. Nonemptiness of $\partial_S K \subset S \cap K$ implies that $z \in \partial_S K$ and $T: \partial_4 K \to S \cap K$ further imlpies that $Tz = z$, which proves the theorem.

If $\partial(S \cap K) \neq 0$, we will show that we arrive at a contradiction. As $S \cap K$ is a closed and convex subset of K, containing more than one element and $T: \partial_S K \to S \cap K$. Thus condition (ii) implies that there exists $x \in \partial_S K$ such that

$$d(x, Tx) = r < \sup_{y \in S \cap K} d(y, Ty). \qquad \ldots(5)$$

Let $P = \{z \in S \cap K : d(z, Tz) \leqslant r\}$ and let $P' = \text{ClCo}\ (TP)$. Then $P' \cap K \neq \phi$. Indeed there exists $x \in \partial_S K$, satisfying eqn. (5), which implies $x \in P$. Therefore

$$d(Tx, T^2 x) \leqslant \frac{d(x, Tx) + d(Tx, T^2 x)}{2}$$

from which it follows that $d(Tx, T^2 x) \leqslant r$, which implies $Tx \in P \subset P'$. Also $x \in \partial_S K$ implies that $Tx \in S \cap K$ that is $Tx \in K$ which further implies that $P' \cap K \neq \phi$.

Next we show that T maps $P' \cap K$ into P'. If $z \in P' \cap K = \{\text{ClCo}\ TP)\} \cap K$, then we have following three possibilities.

(a) $z \in TP$ and $z \in K$. Then there exists $z_1 \in P \subset S \cap K$ such that $Tz_1 = z$. Since $T: S \cap K \to S$, therefore $z = Tz_1 \in S$. Hence $z \in S \cap K$. Therefore

$$d(z, Tz) \leqslant \tfrac{1}{2}\{d(Tz_1, z_1) + d(T^2 z_1, Tz_1)\}.$$

Therefore $d(z, Tz) \leqslant r$ which implies $z \in P$ and $Tz \in TP \subset P'$.

(b) $z \in Co\,(TP)$ and $z \in K$. Then there exists $m \in N$ such that $z \in A_m$ and (as in theorem 2) $d(z, Tz) \leqslant r$. Hence $z \in A_m \cap K$. For $m = 1$, $z = W(Tu, Tv, \lambda)$ for some $u, v \in P \subset S \cap K$ and $\lambda \in [0, 1]$. Since S is convex and $T: S \cap K \to S$, therefore $z = W(Tu, Tv, \lambda) \in S$, which implies $z \in S \cap K$. So by using principle of mathematical induction, it can be easily shown that $z \in A_m \cap K$ implies $z \in S \cap K$ for any m. Hence $z \in P$ and $Tz \in TP \subset P'$.

(c) z is a limit point of Co (TP) and $z \in K$. Then there exists a sequence (z_n) in Co (TP) such that $z_n \to z$. By case (b), $z_n \in P$, therefore $z_n \in S \cap K$ and

$d(z_n, Tz_n) \leqslant r$. Since $S \cap K$ is closed, therefore $z \in S \cap K$. Moreover, continuity of T implies that $Tz_n \to Tz$ and we get

$$d(z, Tz) = \lim_{n\to\infty} d(z_n, Tz_n) \leqslant r.$$

Thus, $z \in P$ and $Tz \in TP \subset P'$.

Hence we find that P' is a closed and convex subset of S such that

$$P' \cap K \neq \phi \text{ and } T : P' \cap K \to P'. \quad \text{Also}$$

$$d(z, Tz) \leqslant r < \sup_{y \in S \cap K} d(y, Ty) \text{ for any } z \in P' \cap K,$$

that is $P' \cap K$ is a proper subset of $S \cap K$. Hence P' is a proper subset of S which is contradiction.

Remark : In view of the remarks in section 1, we find that our Theorems 1, 2 and 3 generalize Theorems 1, 2 and 5 respectively, of Kannan[2].

References

1. M. D. Guay, K. L. Singh, and J. H. M. Whitfield, *Proceedings, Conference on Nonlinear Analysis* (ed. by S. P. Singh and J. H. Burry) Marcel Dekker Vol, 80, 1982, pp. 179–89.
2. R. Kannan, *Proc. Am. Math. Soc.* **38** (1973), 111–18.
3. W. Takahashi, *Kodai Math. Sem. Rep.* **22** (1970), 639–45.
4. L. A. Talman, *Kodai Math. Sem. Rep.* **29** (1973), 62–70.