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SEMI-SYMMETRIC METRIC CONNECTIONS IN A SEMI-RIEMANNIAN MANIFOLD

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It is shown that there exists an interplay between the Riemannian and the semi-Riemannian (Loretzian) geometry with respect to a semi-symmetric metric connection. This paper includes, as a special case, the results of Yano[12] and Imai[4].

1. INTRODUCTION

Yano[13] synthesized the notion of semi-symmetric connection (introduced by Friedmann and Schouten[2]) and a metric connection with torsion[3]. He proved that a Riemannian manifold admits a semi-symmetric metric connection of zero curvature tensor if and only if it is conformally flat[12]. This result was then generalized by Imai[4] for vanishing Ricci tensor of the semi-symmetric metric connection. The objective of the present paper is to further study this topic for semi-Riemannian manifolds endowed with a metric of arbitrary index. Recently, there has been a trend of showing an interplay between the positive definite and indefinite geometry (for example, see O'Neill[7]). The motivation is to present a few fresh ideas on this fruitful relationship.

Consider an n-dimensional orientable semi-Riemannian manifold M with a metric tensor g of arbitrary index and its Levi-Civita connection D. All the geometric objects on M will be assumed sufficiently smooth and arbitrary vector fields will be denoted by X, Y and Z. A linear connection $\tilde{D}$ on M is said to be semi-symmetric metric connection[12] if there exists a 1-form p such that torsion tensor T is given by

$$T(X, Y) = p(Y)X - p(X)Y \text{ and } \tilde{D}g = 0.$$

For such a metric connection it is known that[12]

$$\tilde{D}_X Y = D_X Y + p(Y)X - g(X, Y)P \qquad \ldots(1)$$

where P is a vector field such that $g(P, X) = p(X)$. We denote the curvature tensor, Ricci tensor of type (0.2), Ricci tensor of type (1,1), the scalar curvature and the Weyl conformal curvature tensor of M with respect to D; by K, Ric. Q r and C respectively. A twiddle over them refers to $\tilde{D}$. We know that[12]

$$\tilde{K}(X, Y) = K(X, Y)Z - s(Y, Z)X + s(X, Z)Y - g(Y, Z)SX + g(X, Z)SYs(X, Y) = (D_X p)Y - p(X)\,p(Y) + (1/2)\,p(P)g(X, Y)$$

$$SX = D_X p - p(X)P + (1/2)\,p(P)X \qquad \ldots(2)$$

Note that $C_1^1 \tilde{K} = -C_2^1 \tilde{K}$ and $C_3^1 \tilde{K} = 0$, where C denotes contraction with respect to hth contravariant and kth covariant slots. Hence it is justified to denote $C_1^1 \tilde{K}$ by $\tilde{\text{Ric}}$. C is defined by $C(X, Y)Z = K(X, Y)Z + l(Y, Z)X - l(X, Z)Y + g(Y, Z)\,LX - g(X, Z)\,LY$,

where

$$l = -(1/n - 2)\,\text{Ric} + \{r/2(n-1)(n-2)\}\,g,\ g(LX, Y) = l(X, Y). \qquad \ldots(3)$$

2. Prescribed Ricci Tensor of $\tilde{D}$

In this section we discuss the validity and the implications of the prescription $\tilde{\text{Ric}} = \phi\,\text{Ric}$ where ϕ is a real function on M.

Lemma 1—If $\tilde{\text{Ric}} = \phi\,\text{Ric}$, ϕ a real function on M, then

$$(1 - \phi)\,\text{Ric}(X, Y) = (n-2)\,s(X, Y) + a\,g(X, Y) \qquad \ldots(4a)$$

$$(1 - \phi)\,r = 2(n-1)\,\text{div}\,P + (n-1)(n-2)\,p(P) \qquad \ldots(4b)$$

$$s(X, Y) = (\phi - 1)\,l(X, Y) \qquad \ldots(4c)$$

$$D_X P = p(P)X - (1/2)\,p(P)X - (1 - \phi)\,LX \qquad \ldots(4d)$$

where a is the trace of S and $n \geqslant 3$.

Proof : By hypothesis and appropriate use of contractions we obtain (4ab). Then (4c) and (4d) follow by using (3). Using (4c) in (2) and (3) we state the following theorem :

Theorem 1—If a semi-Riemannian manifold admits a semi-symmetric metric connection such that $\tilde{\text{Ric}} = \phi\,\text{Ric}$. ϕ a real function on M, then for $n > 3$

$$\tilde{K} = \phi K + (1 - \phi)C. \qquad \ldots(5)$$

As an immediate consequence of this theorem we have the following generalized version of the result of Imai[4], and therfore Yano[12].

Corollary 1—If a semi-Riemannian manifold admits a semi-symmetric metric connection with vanishing Ricci tensor, then the curvature tensor of the semi-symmetric metric connection is equal to the Weyl conformal curvature tensor.

Theorem 2—If M is a compact orientable Riemannian manifold without boundary then neither $\tilde{\text{Ric}}$ is identically equal to Ric nor r is equal to zero.

PROOF : Integrating (4b) over M and using Green's theorem[13] we get $\int_M [\{(1-\phi) r/n - 1\} - (n-2)\, g\ (P, P)]\ dv = 0$, let $\tilde{\text{Ric}} = \text{Ric}\ (\phi = 1)$ or $r = 0$. Then $\int_M g\ (P, P)\ dv = 0$ which implies $g\ (P, P) = 0$ and hence $P = 0$, as g is positive definite. But $P=0$ would mean that $\tilde{D} = D$ and hence D would not be semi-symmetric.

The following proposition is immediate from Lemma 1.

Proposition 1— If $\tilde{\text{Ric}} = \phi\ \text{Ric}$, then P is a gradient vector. The congruence of P is geodesic iff $(1 - \phi)\ LP = 0$.

3. AFFINE CONFORMAL MOTIONS

A vector field ξ is said to generate a one-parameter group of conformal motions if

$$(Lg)\ (X, Y) = 2\sigma\ g\ (X, Y) \qquad \ldots(6a)$$

where σ is a smooth function on M and $L = L_\xi$, denotes the operator of Lie-derivative. Every conformal motion must satisfy:

$$(LD)\ (X, Y) = (X\sigma)\ Y + (Y\sigma)\ X - g\ (X, Y)\ \text{grad}\ \sigma. \qquad \ldots(6b)$$

Definition 1—A vector field ξ is said to generate a conformal collineation (Conf C)[10] if (6b) holds.

It follows that every conformal motion is a (Conf C) but the converse is not necessarily true[13]. However, it has been recently proved by the present authors[9] that (6b) is equivalent to

$$(Lg)\ (X, Y) = 2\sigma\ g\ (X, Y) + h\ (X, Y) \qquad \ldots(6c)$$

where $h\ (X,Y)$ is a symmetric (0,2) parallel field on M.

Definition 2—A vector field ξ is called affine conformal[9] if the eqn. (6b) or equivalently (6c) holds.

Definition 3—A vector field ξ is said to generate curvature collineation (CC) if it satisfies[5] $LK = 0$.

Following identities will hold for a (Conf C):

Proposition 2—If M admits a (Conf C) generated by a vector field ξ, then

(a) $(LK)\ (X, Y, Z) = ((D_X\, d\sigma)\ Z)\ Y - ((D_Y\, d\sigma)\ Z)\ X$

$$+ g\ (X, Z)\ D_Y\ \text{grad}\ \sigma - g\ (Y, Z)\ D_X \text{grad}\ \sigma,$$

(b) $(L\text{Ric})(Y, Z) = (\Delta\sigma)\, g(Y, Z) - (n-2)(D_Y d\sigma)Z$,

(c) $Lr = 2(n-1)\Delta\sigma - 2\sigma r - tr(HQ)$,

(d) ξ is a curvature collineation iff D (grad σ) $= 0$,

(e) $(LC)(X, Y, Z) = \{1/n-2\}[g(Y, Z)\ HQX - g(X, Z)\ HQY - h(Y,Z)QX + h(X, Z)\ QY] + \{r/(n-1)(n-2)\}[h(Y, Z)\ X - h(X, Z)\ Y - \{tr.\ HQ/(n-1)(n-2)\}[g(Y, Z)\ X - g(X, Z)\ Y]$

(f) Curvature collineation implies Ricci Collineation [$L\text{Ric} = 0$], where $\Delta\sigma = -$ div (grad σ), $g(HX, Y) = h(X, Y)$ and $g(QX, Y) = R(X, Y)$.

Definition 4—A vector field ξ is said to generate Weyl collineation (WC) if it satisfies[5] $LC = 0$.

Remark 1: The identity (Proposition 2e) for LC does not involve D (grad σ) and therefore remains invariant even when the (Conf C) is restricted to (CC). In particular consider a (Conf C) in a Ricci-flat manifold ($Q = 0$) of dimension $\geqslant 3$. Then. prop. 2b $\Rightarrow$ D (grad σ) $= 0$. Therefore, $LK = 0$. Also $LC = 0$ as $Q = 0$. Hence we state:

Proposition 3—A (Conf C) in a Ricci-flat space is (CC) and (WC). Suppose ξ generates (Conf C) in an Einstein space $\Rightarrow$ Ric $= (r/n)g$. Thus, L Ric $= (r/n)\, Lg$ ($r =$ const). Using Proposition 2f and 2b, we state:

Proposition 4—A (Conf C) ξ in an Einstein space is a (CC) iff it is a motion (isometry). Thus, ξ also generates (WC).

Remark 2: In general, it remains an open problem to find a proper (Conf C) which is also (WC).

Now we study the effect of an affine conformal vector ξ on a semi-symmetric metric connection $\tilde{D}$ invariant under ξ.

Theorem 3—An affine conformal vector field ξ on (M, g), carying a semi-symmetric metric connection $\tilde{D}$, is homothetic ($Lg = Ag$) and $LP = -AP$, $A =$ constant, if and only if $\tilde{D}$ is invariant of the 1-parameter group of transformations generated by ξ.

Proof : From (1), the Lie-derivative along ξ gives

$$(L\tilde{D})(X, Y) = (LD)(X, Y) + (Lp)(Y)\, X - (Lg \otimes P)(X, Y). \qquad \ldots(7)$$

Let $\tilde{L}D = 0$. Substituting (6c) and contracting (7) at Y, we have grad $\sigma = 0$, which means that ξ is affine. Then, contracting (7) at X, $LP = 0$ and hence

$$(Lg \otimes P)(X, Y) = (Lg)(X, Y) P + g(X, Y) LP = 0 \qquad ...(8)$$

Thus, $Lg = Ag$ and $LP = -AP$. Since ξ is affine and conformal, ξ is homothetic and A is const. Converse follows easily.

Corollary 2—Let ξ be a Killing vector field on (M, g) carying a semi-symmetric metric connection $\tilde{D}$. Then $L\tilde{D} = 0 \rightleftharpoons LP = 0$.

Corollary 3—Under the hypothesis of Theorem 3, ξ being homothetic implies that ξ also generates (CC) and (WC).

Finally, we study the effect of an affiine conformal vector field on $\tilde{D}$ with prescribed h.

Theorem 4—If ξ is an affine conformal vector field on (M, g) admitting a semi-symmetric metric connection $\tilde{D}$ with prescription $h = \alpha P \otimes P$; α is a real function, then the congruence of P is a geodesic in M and 2 div $\xi = 2\sigma n + \alpha p(P)$.

PROOF : By hypothesis, $D(\alpha P \otimes P) = 0$. This implies that

$$(X \log \alpha) p(Y) p(Z) + (D_X P)(Y) p(Z) + (D_X p)(z) p(Y) = 0. \qquad ...(9)$$

Contracting the equation (9) with respect to X and Z, we get

$$D_P P + (P \log \alpha + \text{div. } P) P = 0. \qquad ...(10)$$

Thus, by suitable choice of parameter, one can justify that the congruence of P is a geodesic in M. Contracting the equation $Lg = 2\sigma g + \alpha P \otimes P$ we get 2 div $\xi = 2\sigma n + \alpha p(P)$.

Corollary 4—Under the hypothesis of Theorem 4 and $\tilde{\text{Ric}} = \phi$ Ric, P is null iff α satisfies $P \log \alpha + \text{div } P = 0$.

PROOF : Hypothesis of the Theorem 4, and Proposition 1 implies that $(1 - \phi) LP = 0$. This means from Lemma 1 (4d) that :

$$D_P P - (1/2) p(P) P = 0. \qquad ...(11)$$

Thus, the eqns. (10) and (11) $\Rightarrow$ $P \log \alpha + \text{div. } P + (1/2) p(P) = 0$.
Hence $p(P) = 0 \rightleftharpoons P \log \alpha + \text{div. } P = 0$.

Remark 3 : Let M be the 4-dimensional space-time manifold of general relativity. M admits a (CC) $[LK = 0]$ iff. D (gard σ) $= 0$ (Prop. 2) It is known[11] that for ξ to admit a (CC) of an empty space (Ric $= 0$), $Lg = 2\sigma g + \alpha l \otimes l$, if the space is fo Petrov type N, and $Lg = 2\sigma g$ otherwise. Here l is the unique (in direction) real

principal null vector of the Weyl tensor. These physical results are related to our work if we set $p = l$ and Ric $= 0$. For information on the Petrov type, we refer to Kramer *et al.*[6].

4. Geodesics

In this section, we find the conditions under which the geodesics of M with respect to D are preserved by $\tilde{D}$ and then explain their physical meaning.

Theorem 5—A geodesic U of M with respect to D is preserved by $\tilde{D}$ iff either (i) U is null and orthogonal to the integral curves of P or (ii) U is an integral curve of P.

Proof : By hypothesis $D_{U'}U' = 0$, where U' denotes the tangent vector field of U. Then U is a geodesic of M with respect to $\tilde{D}$ iff. $\tilde{D}_{U'}U' = 0$, which, in view of (1), is equivalent to $P(U')\,U' - g(U', U')\,P = 0 \rightleftharpoons$ that either $g(U', U') = g(P, U') = 0$ or $P = U'$ (upto a scalar multiple). This completes the Proof.

Remark 4: The condition (i) has an important physical meaning. Indeed, the geodesics represent light rays in the hypersurface orthogonal to P. To illustrate this point further, consider the space-time V_4 isometrically imbedded as a hypersurface in 5-dimensional manifold (M, g) such that g restricted to V_4 is its Lorentzian metric. Any vector field of M can then be decomposed (uniquely) into its tangential and normal parts. In particular, let P be a normal vector to V_4. Then, a geodesic in M represents a light ray in the space-time. As further study on this application is out of the scope of our paper, we refer to Penrose[8] for similar interpretation of the light rays via *CR*-structures[14].

To interpret the condition (ii), we first recall the following from O'Neill[7]. A variation of a curve U: $[a, b] \to M$ is a 2-parameter mapping f: $[a, b]\, X\, (-\delta, \delta) \to M$ such that for all $(a \leqslant u \leqslant b)$ $U(u) = f(u, 0)$. The u-parameter curves of a variation are called longitudinal and the v-parameter curves transverse. The vector field W on U given by $W(u) = U_V(u, 0)$ is called the variation vector field of U. If U is a geodesic. a vector field X on U that satisfies the equation $D_{U'}\, D_{U'}\, X = K(U', X)\, U'$ is called a Jacobi vector field. It is shown that the variation vector field of a geodesic variation is a Jacobi field.

Corollary 5—Under the hypothesis of theorem 5, if $P = \alpha\, U'$ is a Jacobi field on U then $\alpha = As + B$, where s is the arc length parameter of the congruence of U and A, B are constants.

Proof : $D_{U'}\, D_{U'}\, P = K(U', P)\, U'$ and $P = \alpha\, U'$ implies that $D_{U'}\, [U', \alpha\, U'] + D\,[U', U']\, U' = 0 \Rightarrow U'\, U'\, \alpha = 0$ and hence the solution is $\alpha = As + B$, where s is the arc-length parameter.

Remark 5 : Observe that the Jacobi equation seen as the equation of geodesic deviation has physical interpretation. Indeed, if we consider a geodesic variation as a one-parameter family of freely falling particles, then $D_{U'} D_{U'} P = K(U', P) U'$ represents the Newton's second law, with K in the role of tidal force for a unit mass. Again, limited to the scope of this paper, we refer to O'Neill[7] for further details on the key role of the Jacobi equation in General Relativity.

4. Group Manifolds

We assume $\tilde{K} = 0$ and $\tilde{DT} = 0$. According to Eisenhart[1], a group space has these properties but the converse is not true. However Yano[12] has shown that if a Riemannian manifold M admits a semi-symmetric metric connection satisfying $\tilde{K} = 0$ and $DT = 0$, then M is a group manifold iff M is of constant curvature. While proving this result, he obtained

$$K(X, Y) Z = p(P) \{g(Y, Z) X - g(X, Z) Y\} \qquad ...(12)$$

$$D_X P = p(X) P - p(P) X. \qquad ...(13)$$

It is easy to see that Yano's above result holds also for the semi-Riemannian manifolds. However, for an indefinite metric g, M can be flat whenever P is null. This case is not possible for positive definite g as otherwise $P = 0 \Rightarrow \tilde{D} = D$.

Proposition 5—A compact orientable Riemannian manifold M without boundary can not admit a semi-symmetric metric connection for which M is a group manifold.

Proof : Contracting (13) at X, integrating over M and then applying Green's theorem we get $\int_M p(P)\, dv = 0$ whence $p(P) = 0$ as it is constant. This means that $P = 0$ (since g is positive definite) which is not possible otherwise $\tilde{D} = D$.

We state the following props. for which the proofs are easy

Proposition 6—Let U be a geodesic in a non-flat semi-Riemannian M with a semi-symmetric metric connection for which M is a group manifold. Then the following are equivalent: (a) P is a Jacobi field along U (b) U is a path[1] with respect to $\tilde{D}$ (c) either $P = U'$ (upto a scalar multiple) or $P \perp$ to U' and is null.

Proposition 7—Under the hypothesis of Theorem 3, for a non-flat group manifold M with a semi-symmetric metric connection, an affine conformal vector field is Killing.

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