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# PERTURBATION GEOMETRY FOR MATHIEU'S EQUATION

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The Mathieu Equation

$$\ddot{x} + (\lambda + \mu \cos t)\ x = 0 \qquad \text{...(M)}$$

is a special case of Hill's equation

$$\ddot{x} = Q(t)\ x = 0 \qquad \text{...(H)}$$

(where $Q(t)$ is a function with period $2\pi$), and is important in the theory of nonlinear oscillations. Note that for Mathieu's equation $Q(t)$ is an even function, i. e., $Q(-t) = Q(t)$. The key to understanding the behaviour of the solutions is the stability diagram, i. e., the partition of the $(\lambda, \mu)$ -plane into stable and unstable regions. The transition curves are the frontier separating the stable and unstable regions. The aim of this paper is to throw some light on the structure and origin on these transition curves by regarding them from a geometrical point of view as the inverse images of certain subsets under a smooth map $\Phi$ from the $(\lambda, \mu)$ -plane into the 4-dimensional space of real $2 \times 2$ matrices. Using ideas from singularity theory, we make some observations on how the transition curves may alter as the map $\Phi$ is perturbed.

## INTRODUCTION

The Mathieu equation

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(where $Q(t)$ is a function with period $2\pi$), and is important in the theory of nonlinear oscillations[3,15,23,24,26]. Note that for Mathieu's equation $Q(t)$ is an even function. i. e. $Q(-t) = Q(t)$. The behaviour of solutions of the Mathieu equation has been studied in great detail with particular attention paid to periodic solutions[5,8,20,21,22]. The key to understanding the behaviour is the stability diagram, i. e. the partition of the $(\lambda, \mu)$-plane into stable and unstable regions : these are the sets of points $(\lambda, \mu)$ for which the corresponding equation (M) has all its solutions bounded or has at least one unbounded solution, respectively. The transition curves $\Gamma$ are the frontier separating the stable and unstable regions. See Fig. 1 for a sketch of some of the curves; more

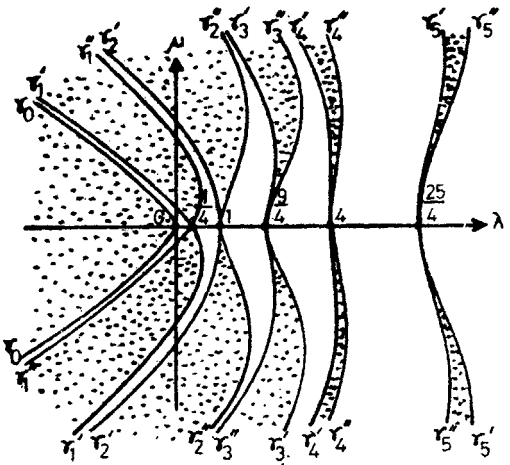


FIG. 1. Stable and unstable regions for the Mathieu equation (the unstable regions are dotted).

accurate diagrams can be found in McLachlan[20], Meixner and Schäfke[22], Nayfeh and Mook[24], and Stoker[26] for example. The key features of interest to us are as described in the following theorem.

*Theorem* 1—(i) the diagram is symmetric with respect to the $\lambda$-axis, and the transition curves $\Gamma$ cross the $\lambda$-axis at the points

$$(\lambda_n, 0) = \left(\frac{n^2}{4}, 0\right) \quad n = 0, 1, 2, \ldots;$$

(ii) apart from the curve $\gamma_0$ through $(\lambda_0, 0) = (0,0)$, the curves form pairs $\gamma_n'$, $\gamma_n''$ having contact of order $n$ at $(\lambda_n, 0)$;

(iii) the curves $\Gamma$ themselves belong to the unstable region, except for the points $(\lambda_n, 0)$, $n \geqslant 1$.

For proofs see the standard texts[20;22] see also e. g. Hale[13], Levy and Keller[19] for (ii), or Stoker[26] for (iii).

The aim of this paper is to throw some light on the structure and origin of these transition curves by regarding them from a geometrical point of view as the inverse images of certain subsets under a smooth map $\Phi$ from the $(\lambda, \mu)$-plane into the 4-dimensional space of real $2 \times 2$ matrices. This contrasts with the thoroughly analytic treatments found in most texts. Our description allows us to distinguish those features of the stability diagram which depend crucially on the Mathieu equation itself from those which would naturally be associated with a wide class of such maps $\Phi$. Moreover, using ideas from singularity theory, we make some observations on how the transition curves may alter as the map $\Phi$ is perturbed: this is a first step in general study of the

response of the stability diagram to arbitrary perturbations of the Mathieu equation. The important question of relating perturbations of $\Phi$ to specific perturbations of the original equation (M) is a separate matter which we shall discuss elsewhere.

## 1. Geometric Description of the Stability Region

### 1.1. *General Theory for Periodic Linear Systems*

Consider a 2-dimensional linear system of ordinary differential equations

$$\left.\begin{aligned} \dot{x} &= c_{11}(t)\,x + c_{12}(t)\,y \\ \dot{y} &= c_{21}(t)\,x + c_{22}(t)\,y \end{aligned}\right\} \qquad \ldots(L)$$

where the $c_{ij}(t)$ are smooth (i.e. infinitely differentiable) real-valued functions of period $2\pi$. Such a system is obtained from Hill's equation, for example, by writing $y = \dot{x}$ so that $(H)$ becomes

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -Q(t)\quad x \end{aligned}\right\}.$$

The solutions to $(L)$ give rise to linear flow in $\mathbf{R}^2$ described by a $2 \times 2$ invertible matrix $A(t)$ in the following way : let $(x(t), y(t)) = (a_{11}(t), a_{21}(t))$ be the unique solution to $(L)$ with initial condition $(x(0), y(0)) = (1, 0)$ and let $(a_{12}(t), a_{22}(t))$ be the unique solution with initial condition (0,1). Then the linearity of $(L)$ and uniqueness of solutions imply that the solution with initial condition $(\zeta, \eta)$ is

$$(x(t), y(t)) = \zeta(a_{11}(t), a_{21}(t)) + \eta(a_{12}(t), a_{22}(t))$$

i. e.

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} \begin{pmatrix} \zeta \\ \eta \end{pmatrix}.$$

Write $A(t)$ for the matrix $(a_{ij}(t))$; then $A(0) = I$ and $A(t)^{-1} = A(-t)$. Note that, except in very special cases, $A(t)$ cannot be found explicitly. Nevertheless, it is the key to understanding the qualitative behaviour of solutions to $(L)$. Since $c_{ij}(t + 2\pi) = c_{ij}(t)$, uniqueness of solutions shows $A(4\pi) = A(2\pi).A(2\pi)$ and in general $A(2m\pi) = A(2\pi)^m$. Hence the long-term dynamics of the solutions to $(L)$ are determined by those of the discrete linear dynamical system obtained by iterating the transformation $A(2\pi) : \mathbf{R}^2 \to \mathbf{R}^2$. In particular, if $A(2\pi)$ has any eigenvalue $\sigma$ with $|\sigma| > 1$ then $(L)$ will have solutions which are unbounded as $t \to +\infty$ (so the origin (0,0) is an unstable equilibrium), while if both eigenvalues $\sigma$ lie inside or on the unit circle in the complex plane then all solutions remain bounded as $t \to \infty$ and the origin is a stable equilibrium, except when $A(2\pi)$ has Jordan form

$$J(\sigma) = \begin{pmatrix} \sigma & 1 \\ 0 & \sigma \end{pmatrix}$$

with $\sigma = \pm 1$ : it is easy to see that iteration of $J(\pm 1)$ gives unbounded solutions.

The above is standard theory ('Floquet theory') for linear systems with periodic coefficients, as found in many texts[3,9,14,18].

Suppose now that the matrix $c(t) = (c_{ij}(t))$ in $(L)$ depends smoothly on $r$ real parameters $(\mu_1, \ldots, \mu_r) = \mu \in \mathbf{R}^r$; then the solution matrix $A(t)$ also depends smoothly on $\mu$ and so we have a smooth map

$$\Phi : \mathbf{R}^r \to (2 \times 2 \text{ matrices}) = M_2 \cong \mathbf{R}^4$$

defined by $\Phi(\mu) = A_\mu(2\pi)$. The geometry of $\Phi$, relative to the decomposition of $M_2$ into subsets corresponding to matrices with various different eigenvalue properties, is important for understanding the dynamics of $(L)$ as the parameters $\mu$ are varied.

*Definition* 1—The stability region in $\mathbf{R}^r$ is the subset $\Phi^{-1}(T_0)$ where

$$T = \{A \in M_2 : \max |\sigma_i| < 1\}$$

with $\sigma_1, \sigma_2$ the eigenvalues of $A$,

and

$$T_0 = \{A \in T : A \text{ does not have Jordan form } J(\pm 1)\}.$$

For the Mathieu equation there are two parameters $(\mu_1, \mu_2) = (\lambda, \mu)$ and the stability region is as described above in the Introduction.

For Hill's equation in general it is easy to check that $d/dt \det A(t) \equiv 0$ and so $\det A(2\pi) = \det A(0) = 1$. Moreover if $Q(-t) \equiv Q(t)$ then

$$a_{22}(2\pi) = a_{11}(2\pi).$$

(see Magnus and Winkler[31], p. 80). These relations give extra constraints for the image of $\Phi$ in $M_2$ and (as we shall see below) force $\Phi(\mathbf{R}^2)$ to intersect $T$ in a very special way, not at all typical of arbitrary maps $\Phi : \mathbf{R}^2 \to M_2$.

### 1.2. *Stratification of the Space of* $2 \times 2$ *Matrices*

Since $A_{\lambda,\mu}(2\pi)$ is invertible, we shall always be working within the open subset $GL(2, \mathbf{R}) \subset M_2$ consisting of invertible matrices. With

$$T = \{A \in M_2 : \max |\sigma_i| < 1\}$$

where $\sigma_1, \sigma_2$, are the eigenvalues of $A$ as above, let

$$S = \{A \in M_2 : \det A = 1\}$$

$$D = \{A \in M_2 : a_{11} = a_{22}\}.$$

Moreover, let $F_i$ $(1 \leqslant i \leqslant 4)$ be the subsets of $M_2$ consisting of matrices described as follows :

$F_1$ : distinct real eigenvalues

$F_2$ : complex (not real) eigenvalues

$F_3$ : Jordan from $\begin{pmatrix} \sigma & 1 \\ 0 & \sigma \end{pmatrix}$

$F_4$ : Jordan from $\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$.

It is an easy matter to describe these sets of matrices explicitly.

If

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} p+q & u+v \\ u-v & p-q \end{pmatrix} \in M_2$$

the sets $T$, $S$ and $F_i$ $(1 \leqslant i \leqslant 4)$ are characterised by the following equalities and inequalities in terms of $p$, $v$ and $w$ :

$T = T_R \cup T_C$ where

$T_R$ : $w^2 \geqslant v^2$ and $\sqrt{w^2 - v^2} \leqslant \min(1 - p, 1 + p)$

$T_C$ : $w^2 < v^2$ and $p^2 + v^2 - w^2 \leqslant 1$;

$S$ : $p^2 + v^2 - w^2 = 1$ ;

$F_1$ : $w^2 > v^2$

$F_2$ $w^2 < v^2$

$F_3$ : $w^2 = v^2 \neq 0$

$F_4$ : $w = v = 0$.

Of course $D$ is given by $q = 0$, in which case $w^2 = u^2$.

For fixed $w \neq 0$ the set $T_R$ is the closure of the region lying inside the rectangle defined by the lines $v = \pm w$, $p = \pm 1$ and outside the two circles $v^2 + (1 \pm p)^2 = w^2$. Also $T_C$ is the region outside the lines $v = \pm w$ but inside or on the circle $p^2 + v^2 = 1 + w^2$. These are illustrated in Fig. 2.

The boundary of $T$, namely the set

$$\Sigma = \{A \in M_2 : \max | \sigma_i | = 1\}$$

plays a central role in our investigations, as it must do in any problem of stability for families of linear discrete dynamical systems. In the terminology of Arnol'd[2,4] it is

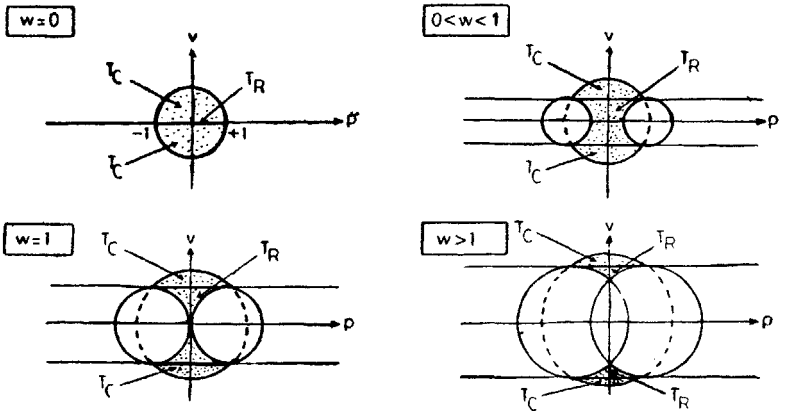


FIG. 2. The set $T = T_R \cup T_C$ in $M_2$.

the boundary of stability (stability boundary) for the 4-parameter family of matrices $M_2$. Its geometrical structure is seen easily from the description of $T = T_R \subset T_C$ above.

We might expect that the transition curves are simply $\Phi^{-1}(\Sigma)$; however, this is not the case because of the special constraints on $\Phi$ mentioned above in §1.1.

The set $S$ is represented by a smooth quadric surface (hyperboloid of one sheet) in $(p, v, w)$-space; it cuts the $p$-axis at $(\pm 1, 0, 0)$ and its tangent planes at these points are $P^{\pm}$: $p = \pm 1$. Let $S_i$ denote $S \cap F_i$ $(1 \leqslant i \leqslant 4)$; then $S_4 = \{(\pm 1, 0, 0)\}$, and $S_3 = S_3^+ \cup S_3^-$ where $S_3^+$ is a pair of lines $S \cap P^+$ and similarly for $S_3^- = S \cap P^-$. The part of $S$ lying between $P^+$ and $P^-$ is $S_2$, and $S_1$ is the rest of $S$ consisting of the two components $S_1^{\pm}$ where $p > 1, p < -1$. Thus we have a decomposition of $S$ into submanifolds. i. e. a stratification of $S$, in which the strata $S_1, S_2, S_3, S_4$ have codimension (i. e. dim $S$-dim $S_i$) equal to 0, 0, 1, 3 respectively : remember here that the variable $w$ represents $u$, $q$ together. See Fig. 3 for a sketch of this stratification.

If $w$ is replaced by $u$ then Fig. 3 represents at the same time the stratification of $S \cap D$ by the manifolds $S_i \cap D$; here, though, $S \cap D$ is 2-dimensional, and $S_4 \cap D = \{(\pm 1, 0, 0)\}$ has codimension 2 in $S \cap D$.

Observe that $\Sigma \cap S = T \cap S = S_2 \cup S_3 \cup S_4$, and this provides a natural stratification for part of $\Sigma$. We shall write $S_i$ as $\Sigma_i$ $(i = 2, 3, 4)$ when thinking of them as strata for $\Sigma$. The rest of $\Sigma$ consists of the intersection of the two cones

$$C^{\pm} : v^2 + (1 \pm p)^2 = w^2$$

which forms a stratum $\Sigma_3'$ of codimension 2 in $M_2$ (represented in $(p, v, w)$-space by

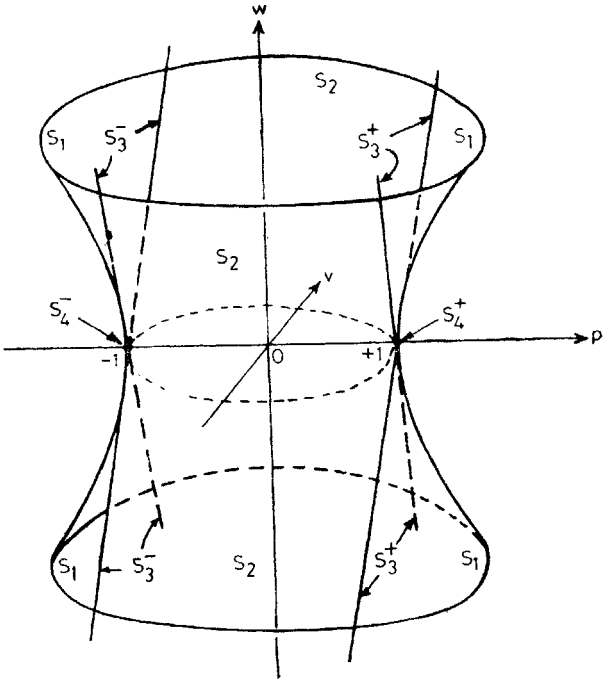


FIG. 3. The stratification $\{S_i\}$ of $S$ in $M_2$.

the hyperbola $p = 0$, $(w^2 - v^2 = 1)$, together with an open subset $\Sigma_2'$ of $C^+ \cup C^-$, of codimension 1 in $M_2$. Both $\Sigma_2'$ and $\Sigma_3'$ belong to the boundary of $T_R$.

Observe that at each point of $\Sigma$ except $(\pm 1, 0,0)$ the local structure of $\Sigma$ fits with the description given by Arnol'd[2,4] [(2) p. 85 [4]; p 256] for the local structure of the boundary of stability for a generic (i. e. typical) family of matrices. Along $\Sigma_2$ we have a ridge or corner of 'simple passing' type, while $\Sigma_3$ is a "Jordan 2-block" corner. Note that Arnol'd deals with the problem of maximizing the real parts of eigenvalues, while we are concerned with their absolute values, but the theory is essentially the same : see the end of § 30*F* in Arnol'd[4], or the paragraph preceding § 4.1 in Arnol'd[2] (In Arnol'd[4] the case $|\sigma| = 1$ is excluded, but (compare Arnol'd[2]) this seems to be an error and the exclusion should refer to $\sigma = 0$.)

## 2. Transversality of $\Phi$ to the Strata

We now turn to the map $\Phi : \mathbf{R}^2 \to M_2 \cong \mathbf{R}^4$ and its configuration relative to $\Sigma$ and to the stratification $\{S_i\}$ of $S$. We investigate how 'special' $\Phi$ is by using the idea of general position or transversality with respect to the various strata.

2.1. *Basic Ideas of Transversality*

Let $F: \mathbf{R}^m \to \mathbf{R}^p$ be a smooth map, and let $W$ be a smooth manifold in $\mathbf{R}^p$. (For example, we think of $W$ as a (piece of) 2-dimensional surface or curve in $\mathbf{R}^3$, or perhaps in $\mathbf{R}^4$.)

For each $x$ in $\mathbf{R}^m$ let $DF(x)$ denote the $p \times m$ matrix of the first partial derivatives of the components $F_i$ $(1 \leqslant i \leqslant p)$ with respect to $x_j$ $(1 \leqslant j \leqslant m)$ evaluated at $x$: $DF(x) = \left( \frac{\partial F_i}{\partial x_j}(x) \right)$.

*Definition* 2—The map $F$ is transverse to $W$ at $x$ if either $F(x)$ is not in $W$, or $F(x)$ is in $W$ and the range of $DF(x)$ together with the tangent space $T_yW$ to $W$ at $y = F(x)$ span the whole of $\mathbf{R}^p$,

i. e.

$$\text{range } DF(x) + T_yW = \mathbf{R}^p.$$

If $F$ is transverse to $W$ at every $x \in \mathbf{R}^n$ we simply say $F$ is transverse to $W$.

Thus $F$ is transverse to $W$ if, whenever $F(\mathbf{R}^m)$ meets $W$, the local linear approximations to $F(\mathbf{R}^m)$ and to $W$ are in general position in $\mathbf{R}^p$.

As well as capturing the idea of general position' in a precise way, transversality is closely related to the notion of structural stability, or persistence of geometric structures under small perturbations of the data. If $F: V \to M$ is transverse to a manifold $W$ which is a closed subset of $M$, then the same holds for all $G: V \to M$ which are sufficiently close (in their derivatives also) to $F$; moreover the set $G^{-1}(W)$ is qualitatively the same as $F^{-1}(W)$, and in fact $G^{-1}(W)$ can be moved back to coincide with $F^{-1}(W)$ by performing a smooth deformation of $V$. This applies also in more general cases when $W$ is not closed in $M$, provided we restrict attention to $G^{-1}(W')$ where $W'$ lies outside some neighbourhood of $\overline{W} \setminus W$. We shall use this below when investigating structural stability of the transition curves $\Gamma$ for the Mathieu equation.

For further discussion of transversality see the references mentioned above, or other books[6,12] on differential topology.

The important Transversality Theorem of Thom states that the property of transversality to $W$ is a typical or generic property for maps $F: \mathbf{R}^m \to \mathbf{R}^p$. Formally, this is expressed in terms of a suitable topology (Whitney topology) on the space $C^1(\mathbf{R}^m, \mathbf{R}^p)$ of $C^1$ maps $\mathbf{R}^m \to \mathbf{R}^p$. We shall not pursue the details here but refer to texts on differential topology[11,16,17] ([11]Ch. 2, [16] Chs 2, 3, [17]Chs. 2,3). Note that when the domain is noncompact (as with $\mathbf{R}^m$ here) extra care is needed to take account of behaviour at infinity: a perturbation $G$ of $F$ is not considered 'small' unless $G(x) - F(x)$ and $DG(x) - DF(x)$ approach zero as $|x| \to \infty$.

It is a consequence of the Implicit Function Theorem, using local coordinates $(y_1, \ldots, y_p)$ near $y$ in which $W$ is given by $y_{k+1} = \ldots = y_p = 0$ (where $k = \dim W$),

that if $F$ is transverse to $W$ and $F^{-1}(W)$ is non empty then $F^{-1}(W)$ is a manifold in $\mathbf{R}^m$ whose codimension is the same as that of $W$ in $\mathbf{R}^p$, namely

$$m - \dim F^{-1}(W) = p - k.$$

In particular, if $p - k > m$ then $F^{-1}(W)$ must be empty, i. e. if $F$ is transverse to $W$ in this case then $F(\mathbf{R}^m)$ avoids $W$ altogether. This formalizes the familiar idea that "$q$ equations in $m$ unknowns will in general have a solution space with $m - q$ degrees of freedom"; here $q = p - k$.

The definition extends easily to maps $F: \mathbf{R}^m \to M$ where $M$ is an arbitrary smooth manifold of dimension $p$ and $W$ is a submanifold of $M$; in this case the key relation is

$$\text{range } DF(x) + T_yW = T_yM.$$

Also $\mathbf{R}^m$ may be replaced by a smooth manifold $V$; then $DF(x)$ is a linear map $T_x V \to T_y M$.

2.2. *Transversality for* $\Phi$

(a) Recall from 1.1., 1.2 above that $\Phi : \mathbf{R}^2 \to M_2 \cong \mathbf{R}^4$ satisfies

$$\Phi(\mathbf{R}^2) \subseteq S \cap D.$$

In particular $\Phi(\mathbf{R}^2) \subset S_1 \cup S_2 \cup S_3 \cup S_4$ and since each $S_i$ has codimension $\geqslant 1$ in $M_2$ it is easy to see on purely dimensional grounds :

*Lemma* 1— $\Phi : \mathbf{R}^2 \to M_2$ is not transverse to any of the strata $S_1, S_2, S_3, S_4$ of $S$.

This says at the same time that $\Phi$ is not transverse to the strata $\Sigma_2, \Sigma_3, \Sigma_4$, of $\Sigma$. However, $\Phi$ is (technically) transverse to $\Sigma_2'$, $\Sigma_3'$ since $\Phi(\mathbf{R}^2)$ does not meet these two strata of $\Sigma$.

Since $\Phi(\mathbf{R}^2) \subset S$ and $T \cap S = \Sigma \cap S$ we have

$$\Phi(\mathbf{R}^2) \cap T = \Phi(\mathbf{R}^2) \cap S,$$

i. e. $\Phi^{-1}(T) = \Phi^{-1}(\Sigma)$, which is not what we could expect for a typical map from $\mathbf{R}^2$ into $M^2$. Thus the transition curves $\Gamma = \{\gamma_0; \gamma_n', \gamma_n'', n \geqslant 1\}$ do not constitute $\Phi^{-1}(\Sigma)$. Since

$$S \cap (M_2 \setminus T) = S_1$$

the transition curves $\Gamma$ must however be the inverse image under $\Phi$ of the boundary of $S_1$ in $S$; thus

*Proposition* 1— $\Gamma = \Phi^{-1}(S_3 \cup S_4)$.

In fact we know by Theorem 1 (iii) that $\Phi^{-1}(S_4) = \Phi^{-1}(\pm 1, 0,0)$ is the set of points

$$\Lambda = \{(\lambda_n, 0) : n \geqslant 1\}$$

on the $\lambda$-axis, while $\Phi^{-1}(S_3)$ is $\gamma_0$ together with the rest of $\Gamma$ lying off the $\lambda$-axis.

(b) Next we regard $\Phi$ as a map $\Phi : \mathbf{R}^2 \to S$ and look again at transversality with respect to $S_i$ $(1 \leqslant i \leqslant 4)$ in $S$. Since $S_1$, $S_2$ are open sets (codimension 0) in $S$ and $S_4$ has codimension 4 we can state immediately on dimensional grounds :

*Lemma* 2—For $\Phi : \mathbf{R}^2 \to S$ we have

(i) $\Phi$ is transverse to $S_1$ and $S_2$ in $S$;

(ii) $\Phi$ is not transverse to $S_4$ in $S$.

It remains to consider how $\Phi$ meets $S_3$.

*Lemma* 3— $\Phi : \mathbf{R}^2 \to S$ is transverse to $S_3$.

PROOF : Let $\pi : S \to \mathbf{R}$ denote the restriction to $S$ of the projection of $(p,q,u,v)$ -space $\mathbf{R}^4$ onto the $p$-axis. We have

$$\pi^{-1}(+1) = S_2^+ \cup S_4^+ \,, \quad \pi^{-1}(-1) = S_3^- \cup S_4^-$$

and, moreover, the points of $S_3^{\pm}$ are regular points of $\pi$ (the derivative of $\pi$ does not vanish there). Since $S_3$ has codimension 1 in $S$, in order to show that $\Phi$ is transverse to $S_3$ it will suffice to show that the function

$$f = \pi \circ \Phi : \mathbf{R}^2 \to \mathbf{R}$$

is regular at points $(\lambda, \mu) \in \Phi^{-1}(S_3)$. Now for fixed $\mu$ the function

$$\Delta_\mu = 2f(\cdot, \mu) : \mathbf{R} \to \mathbf{R}$$

(often called the discriminant for Mathieu's equation) has been well studied in the literature. Indeed, $2f(\lambda, \mu) = \text{trace } A_{\lambda,\mu}(2\pi)$ and the stability regions for $\mu \neq 0$ are characterized by $|\Delta_\mu| < 2$. The following is known :

*Lemma* 4—The zeros of $\Delta_\mu - 2$ and $\Delta_\mu + 2$ are simple zeros,

i. e. if $\Delta_\mu(\lambda') = \pm 2$ then $\dfrac{d}{d\lambda}\Delta_\mu(\lambda') \neq 0$.

PROOF : [See Magnus and Winkler[21], 2.5] or [Coddington and Levinson[9], Lemma 3.1]. These proofs show that if $\Delta_\mu(\lambda') \pm 2 = \dfrac{d}{d\lambda}\Delta_\mu(\lambda') = 0$ then the matrix $A(t) = A_{\lambda,\mu}(t)$ (in our terminology) has to be diagonal for all $t$,

i. e. $a_{21}(t) = a_{12}(t) = 0$. However, for $(\lambda, \mu')$ belonging to $\Phi^{-1}(S_3)$ this is not the case when $t = 2\pi$, by definition of $S_3$. This result that points $(\lambda, \mu) \in \Phi^{-1}(S_3)$ are regular points of $\Delta$, which proves Lemma 3.

(c) Finally, let us regard $\Phi$ as a map $\Phi : \mathbf{R}^2 \to S \cap D$ and consider transversality of $\Phi$ to the strata $S_i \cap C$ in $S \cap D$ : recall that $S \cap D$ is the 2-dimensional quadric surface (hyperboloid) $p^2 + v^2 - u^2 = 1$ in $(p, u, v)$-space. Since $S_1 \cap D$, $S_2 \cap D$ are open sets in $S \cap D$ we automatically have transversality for those strata, and the argument of Lemma 3 shows transversality of $\Phi$ to $S_3 \cap D$. Note now that on dimensional grounds it is quite possible for $\Phi : \mathbf{R}^2 \to S \cap D$ to be transverse to $S_4 \cap D = \{(\pm 1, 0, 0)\}$ since each point $(\pm 1, 0, 0)$ has codimension 2 in $S \cap D$. Transversality here is just the condition that at each point $(\lambda_n, 0)$ of $\Lambda$ the derivative $D\Phi(\lambda_n, 0)$ should have rank 2, i. e. the two $2 \times 2$ matrices

$\frac{\partial}{\partial \lambda} A_{\lambda_n, 0}(2\pi)$, $\frac{\partial}{\partial \mu} A_{\lambda_n, 0}(2\pi)$ should be linearly independent. By the Inverse Function Theorem this is equivalent to $\Phi$ being a local diffeomorphism from a neighbourhood of $(\lambda_n, 0)$ in $\mathbf{R}^2$ to a neighbourhood of $(\pm 1, 0, 0)$ in $S \cap D$.

We know in advance, however, that this transversality must fail for $n \geqslant 2$, because it would imply that the two curves $\gamma_n'$ , $\gamma_n''$ meet at nonzero angle at $(\lambda_n, 0)$ just as $(S_3 \cup S_4) \cap D$ consists of two lines meeting at nonzero angle in $S \cap D$, and this contradicts Theorem 1 (ii).

It is easy to verify the failure of transversality by computing the derivatives of $\Phi$ at $(\lambda_{n,0})$ explicitly. The equation (M) with $\mu = 0$ is easy to solve and gives

$$A_{\lambda, 0}(2\pi) = \begin{pmatrix} \cos 2\pi\alpha\, \alpha^{-1} & \sin 2\pi\alpha \\ -\alpha \sin 2\pi\alpha & \cos 2\pi\alpha \end{pmatrix}$$

for $\lambda = \alpha^2 > 0$; then we find

$$\frac{\partial \Phi}{\partial \lambda}\begin{pmatrix} \lambda_n, 0 \end{pmatrix} = \frac{d}{d\lambda} A_{\lambda_n, 0}(2\pi) = (-1)^n \pi \begin{pmatrix} 0 & 4n^{-2} \\ -1 & 0 \end{pmatrix}.$$

For $\frac{\partial \Phi}{\partial \mu}(\lambda_n, 0)$ more work is needed. We apply the standard method of writing a solution to (M) as

$$x(t) = x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \ldots$$

and substituting into (M). We obtain a sequence of equations for the coefficients of ascending powers of $\mu$ :

$$\begin{aligned} &\ddot{x}_0 + \lambda x_0 = 0 \\ &\ddot{x}_1 + \lambda x_1 = -x_0 \cos t \\ &\ddot{x}_2 + \lambda x_2 = -x_1 \cos t \end{aligned} \qquad \ldots(E)$$

By successively solving these with appropriate initial conditions we obtain the matrix $A_{\lambda, \mu}(t)$ in the form

$$A_{\lambda,\mu}(t) = A^0_\lambda(t) + \mu A^1_\lambda(t) + \mu^2 A^2_\lambda(t) + \ldots$$

Then

$$\frac{\partial \Phi}{\partial \mu}(\lambda_n, 0) = A^1_{\lambda_n}(2\pi).$$

We find[1] that

$$A^1_0(2\pi) = \begin{pmatrix} 0 & 4\pi \\ 0 & 0 \end{pmatrix}$$

$$A^1_{1/4}(2\pi) = \begin{pmatrix} 0 & 2\pi \\ \pi/2 & 0 \end{pmatrix}$$

and for all $(\lambda, 0) = (\alpha^2, 0)$ with $\alpha \neq 0, \frac{1}{2}$ we have

$$A^1_\lambda(2\pi) = -2(4\alpha^2 - 1)^{-1} \sin 2\pi\alpha \begin{pmatrix} 0 & \alpha^{-1} \\ \alpha & 0 \end{pmatrix}.$$

In particular, $A^1_{\lambda_n}(2\pi) = 0$ for $n \geqslant 2$.

We see then that the matrices $\frac{\partial \Phi}{\partial \lambda}(\lambda_n, 0)$, $\frac{\partial \Phi}{\partial \mu}(\lambda_n, 0)$ are linearly independent for $n = 0, 1$ but $\frac{\partial \Phi}{\partial \mu}(\lambda_n, 0) = 0$ for $n \geqslant 2$. Hence, reverting to $(p, u, v)$ coordinates in $D$, we may conclude:

*Lemma* 5—The map $\Phi : \mathbf{R}^2 \to S \cap D$ is transverse to $(-1, 0, 0)$ at $(\lambda, \mu) = (\frac{1}{4}, 0)$ but is not transverse to $S_4$ at $(\lambda_n, 0)$ for $n \geqslant 2$.

(Recall that $\Phi(0, 0)$ belongs to $S_3$ and not $S_4$)

*Note*: From $A^{0,1}_\lambda(2\pi)$ alone we can calculate the 2-jet (Taylor series up to terms of degree 2) of the function

$$f(\lambda, \mu) = \tfrac{1}{2}\Delta(\lambda, \mu) = \text{trace } A_{\lambda,\mu}(2\pi)$$

at $(\lambda, \mu) = (\lambda_n, 0)$, $n > 1$. Since $f(\lambda_n, 0) = (-1)^n$ and $f^2 - a_{12}a_{21} = \det A = 1$, where $A_{\lambda,\mu}(2\pi) = A = (a_{ij})$, we have

$$f = (-1)^n (1 + a_{12}a_{21})^{1/2}.$$

Now with $A^{0,1}_\lambda(2\pi)$ found above we have

$$A_{\lambda,\mu}(2\pi) = \begin{pmatrix} c & \alpha^{-1}s \\ -\alpha s & c \end{pmatrix} - \mu.2(4\lambda - 1)^{-1} s \begin{pmatrix} 0 & \alpha^{-1} \\ \alpha & 0 \end{pmatrix} + O(\mu^2)$$

where $\alpha^2 = \lambda > 0$ and $c, s = \cos 2\pi\alpha, \sin 2\pi\alpha$, so for $\lambda = \frac{n^2}{4} + \zeta$

$$a_{12}\, a_{21} = \sin^2 2\pi\ \zeta\,(4\mu^2\,((n^2 - 1) + 4\zeta)^{-2} - 1) + O\,(3)$$

$$= \begin{cases} -\sin^2 2\pi\zeta + O\,(3)\ , & n \neq 1 \\ \sin^2 2\pi\ \zeta\left(\frac{\mu^2}{4\zeta^2} - 1\right) + O\,(3), & n = 1 \end{cases}$$

where $O\,(r)$ denotes terms in $\zeta$, $\mu$ of order at least $r$. Hence

$$f\,(\lambda_n + \zeta, \mu) = \begin{cases} (-1)^n\,(1 - 2\pi^2\,\zeta^2 + O\,(3)), & n \neq 1 \\ -\,(1 + 2\pi^2\left(\frac{\mu^2}{4} - \zeta^2\right) + O\,(3)), & n = 1. \end{cases}$$

This shows in particular that the transition curves $\gamma_1'$ , $\gamma_1''$ which are the solutions of $f\,(\lambda_1 + \zeta, \mu) = 0$ in $(\zeta, \mu)$ -space, have the form

$$\zeta = \pm\, \tfrac{1}{2}\,\mu + O\,(\mu^2)$$

which agrees with the descriptions of $\gamma_1'$ , $\gamma_1''$ given elsewhere in the literature.

Warning : many authors consider Mathieu's equation in the form $\ddot{x} + (\lambda + 2\mu \cos 2t)\ x = 0$, so that translation has to be made to our notation.)

## 3. Structural Stability and Instability of the Transition Curves

We next aim to relate the transversality results obtained above to the question of structural stability of the transition curves, i. e. the persistence or otherwise of these curves $\Gamma = \Phi^{-1}\,(S_3 \cup S_4)$ under small perturbations of the map $\Phi : \mathbf{R}^2 \to M_2$. Using the remarks in § 2.1 and the results of Lemmas 2 (i), 3 we can state the following theorem.

*Theorem* 2— (i) Let $V$ be a given neighbourhood of $\Lambda_1 = \{(\lambda_n, 0): n \geqslant 1\}$ in $\mathbf{R}^2$. Then there is a neighbourhood $\mathscr{U}$ of $\Phi$ in the space $C^1\ (\mathbf{R}^2, S)$ of $C^1$ maps $\mathbf{R}^2 \to S$ such that if $\Psi$ belongs to $\mathscr{U}$ then outside $V$ the transition curves $\Gamma' = \Psi^{-1}\,(S_3)$ for $\Psi$ are merely perturbed versions of $\Gamma$.

(ii) A typical map in $C^1\ (\mathbf{R}^2,\ M_2)$ will avoid the $p$-axis, and the stability region $\psi^{-1}\,(T_0)$ will (if nonempty) consist of a region in $\mathbf{R}^2$ bounded by transition curves $\Psi^{-1}\,(\Sigma_2)$ $\Psi^{-1}\left(\ \Sigma_2'\ \right)$ which wherever they meetdo so with nonzero angle at points of $\Psi^{-1}\ (\Sigma_3 \cup \Sigma_3')$.

Thus if $\Psi$ is constrained to map $\mathbf{R}^2$ into $S$ and is sufficiently close to $\Phi$, the transition curves persist away from the $\lambda$-axis. However, if $\Psi$ is not so constrained then typically the diagram of transition curves $\Gamma$ will break completely and the stability region for $\Psi$ will take a form whose local structure is as described by Arnol'd[2'4] [2] §4.2; [4] § 30*K*].

We go on to discuss what can happen to the transition curves close to the λ-axis when $\Psi(\mathbf{R}^2) \subset S$. For this we need some results from singularity theory on deformations of maps near singular points.

## 4. Deformations of Singularities

Let $f(x, y)$ be a $C^1$ function of two real viables, with $f(0, 0) = 0$ but $\frac{\partial f}{\partial x}(0,0)$, $\frac{\partial f}{\partial y}(0, 0)$ not both zero, so the origin is a regular point for $f$. Equivalently, $f$ is transverse to $\{0\}$ at $(0, 0)$. The Implicit Function Theorem tells us that in some neighbourhood of the origin the zero set $f^{-1}(0)$ (i.e. the set of solutions $(x,y)$ to $f(x, y) = 0$) is a $C^1$ curve passing through $(0, 0)$. Moreover, the same will be true for any other function $g$ sufficiently close in the $C^1$ sense to $f$, because then also $\left(\frac{\partial g}{\partial x}(0, 0), \frac{\partial g}{\partial y}(0, 0)\right) \neq (0, 0)$.

If, however, $\frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = 0$ then the origin is a singular point or critical point for $f$, and the structure of $f^{-1}(0)$ may indeed change significantly when $f$ is perturbed. For example, if $f(x, y) \equiv x^2 + y^2$ and $g(x, y) = f_c(x, y) \equiv x^2 + y^2 + c$ then $f_c^{-1}(0)$ changes from a circle to a point and then to the empty set as the parameter $c$ passes through 0 from below.

It is clear geometrically (and not very hard to prove) that the 1-parameter family $f_c$ captures everything that can happen qualitatively to the zero set as the function $f$ is perturbed in any way at all, provided that the perturbation is small enough both in the function and its derivatives.

Singularity theory (in particular, elementary catastrophe theory) allows analogous statements to be made for more degenerate types of critical point. It shows how certain $r$-parameter families $f_a$ (where $a$ denotes a multiparameter in $\mathbf{R}^r$) have a property of universality, in that they represent all possible perturbations of the function $f_0$ -at least if the perturbations are sufficiently small (in derivatives also) and we work all the time in some neighbourhood of the original critical point and of 0 in $\mathbf{R}^r$. Such a family is called a universal unfolding or universal deformation for $f_0$. Sometimes also universal is replaced by versal in order not to suggest uniqueness.

The result that we shall wish to use is the following :

*Theorem* 3—Let $f(x, y) \equiv x^2 \pm y^m$. Then an example of a universal unfolding for $f$ is the $(m - 1)$-parameter family

$$F(c, a, \ldots, a_{m-2}; x, y) \equiv c + x^2 + a_1 y + a_2 y^2 + \ldots + a_{m-2} y^{m-2} \pm y^m.$$

Specifically, any $g$ sufficiently close to $f$ can be expressed near $(0, 0)$ as

$$g(x, y) \equiv F(c, a, \ldots a_{m-2}; x', y')$$

for a suitable coordinate transformation $(x, y) \to (x', y')$ and appropriate values of the parameters $c, a_1, \ldots a_{m-2}$.

Proofs of this fact and general background material on singularity theory can be found in the references Brocker and Lander[7], Gibson[10], Golubisky and Guillemin[11] Poston and Stewart[25], Zeeman[27]. Note that as we are interested in the values of a function rather than its critical points, our unfoldings include constant terms.

For small values of $m$ it is feasible to sketch how the form of the set of solutions $(x, y)$ to $F = 0$ changes as $(c, a, \ldots, a_{m-2})$ moves in parameter space $\mathbf{R}^{m-1}$. For $m = 2$ we have the familiar pictures shown in Fig. 4, and for $m = 3$ a cusp in parameter space $\mathbf{R}^2$ separates the two main types of zero set as seen in Fig. 5. For $m = 4$

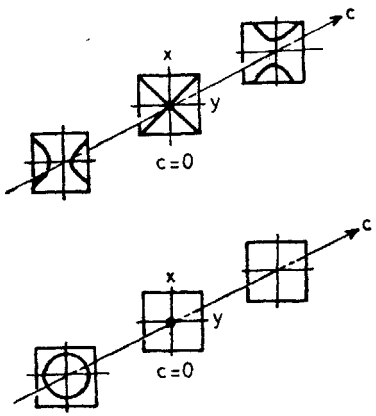


FIG. 4. Changes in zero set of $c + x^2 \pm y^2$ as $c$ varies.

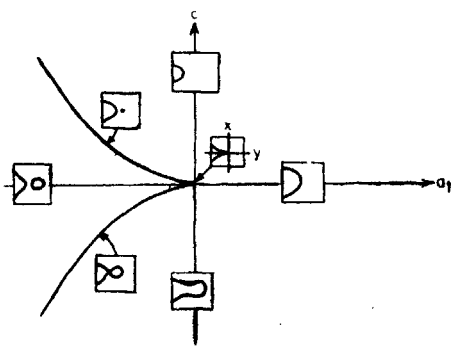


FIG. 5. Changes in the zero set of $c + x^2 + a_1 y + y^3$ as $(c, a_1)$ varies.

the swallowtail separates parameter space $\mathbf{R}^3$ into three regions giving different types of zero set for $F$, as shown in Fig. 6 for the negative choice of sign.

A similar theory of versal unfoldings applies to singularities of maps $\Phi: \mathbf{R}^n \to \mathbf{R}^p$ although for $p > 1$ they are more usually called versal deformations. We shall be interested only in $(n, p) = (2, 2)$ or $(2,3)$, and in a very small part of the theory. The definition of $r$-parameter versal deformation goes as for the case $p = 1$ above, except that now we must also include a coordinate transformation (diffeomorphism) for $\mathbf{R}^p$. Recall that we work all the time on some neighbourhood of the origin in $\mathbf{R}^r \times \mathbf{R}^n$.

*Definition* 3—The map $\Phi: \mathbf{R}^n \to \mathbf{R}^p$ is stable (at the origin) if it is its own versal unfolding.

This implies that any small deformation of $\Phi$ can be converted back into $\Phi$ itself by coordinate changes in $\mathbf{R}^n$ and $\mathbf{R}^p$.

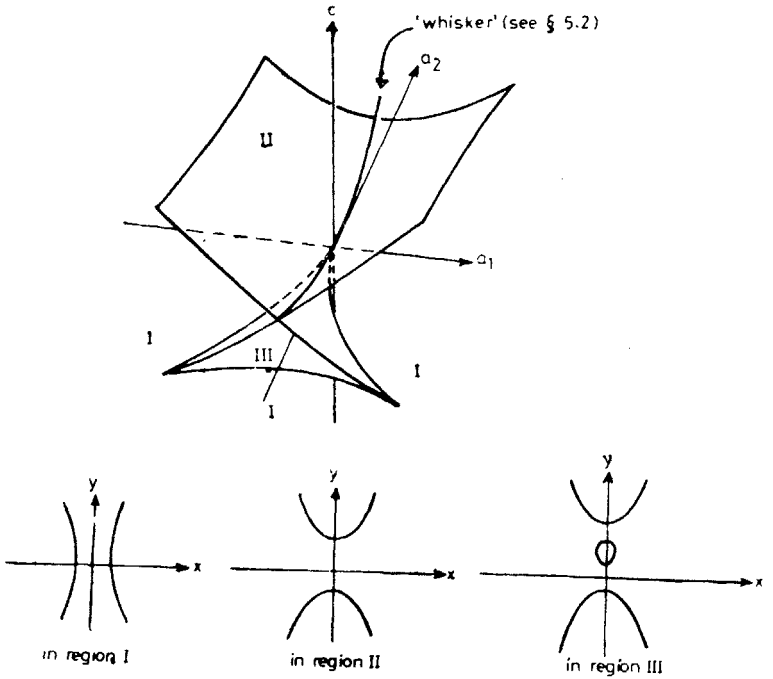


FIG. 6. Changes in the zero set of $c + x^2 + a_1 y + a_2 y^2 - y^4$ as $(c, a_1, a_2)$ varies.

Stable maps in low dimensions have been completely classified :

*Theorem* 4 (Whitney)—(i) A map $\Phi : \mathbf{R}^2 \to \mathbf{R}^2$ with $\Phi(0,0) = (0, 0)$ is stable at (0,0) if and only if it can be written in appropriate local coordinates near (0, 0) as one of three types :

(a) $\Phi(x, y) = (x, y)$ Nonsingular

(b) $\Phi(x, y) = (x, y^2)$ Fold

(c) $\Phi(x, y) = (x, xy + y^3)$ Cusp

(ii) A map $\Phi\ \mathbf{R}^2 \to \mathbf{R}^3$ with $\Phi(0, 0) = (0, 0, 0)$ is stable at (0, 0) if and only if it can be written in appropriate local coordinates near (0, 0) as one of two types :

(*a*) $\Phi(x, y) = (x, y, 0)$ Nonsingular

(b) $\Phi(x, y) = (x, xy, y^2)$ Umbrella.

Moreover, to decide whether a given $\Phi$ is one of these types it suffices to find coordinates in which $\Phi$ has the stated form plus terms of degree at least 2 (for (i) (a), (ii) (a)), 3 (for (i) (b), (ii) (b)) or 4 (for (i) (c)).

For proofs see Gibson[10], Golubitsky and Guillemin[11]; for illustrations Fig. 7.

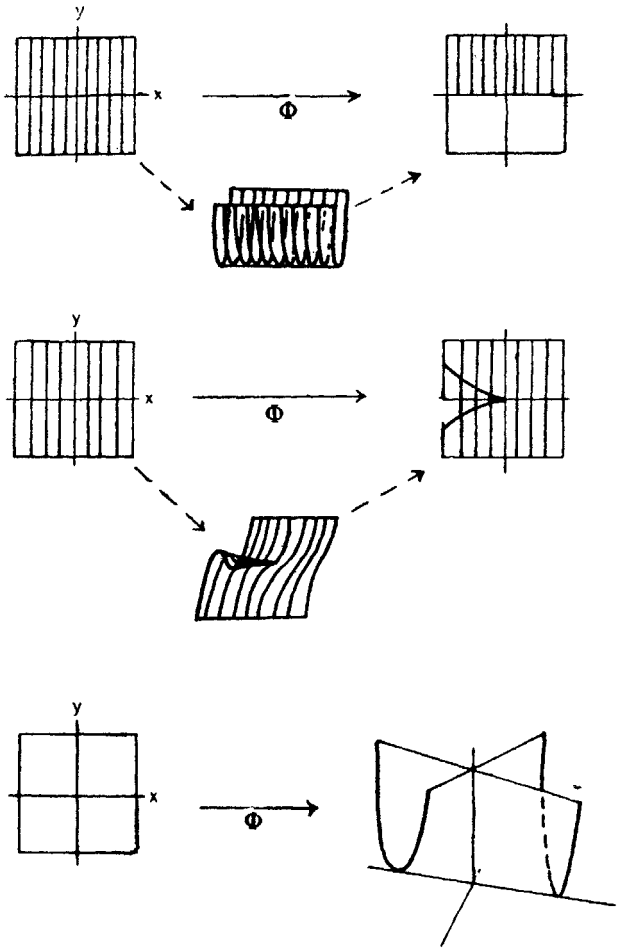


FIG. 7. Fold, cusp and umbrella.

Armed with this theory of versal deformations and stability, we look again in the next section at the maps $\Phi : \mathbf{R}^2 \to S \cap D$, $\Phi : \mathbf{R}^2 \to S$ and the function $f = \pi \circ \Phi$: $\mathbf{R}^2 \to \mathbf{R}$.

## 5. Structural Instability of the Transition Curves close to the $\lambda$-axis

### 5.1. *Singularities of* $\Phi$ *at* $(\lambda_n, 0)$, $n > 2$

To study the singular behaviour of $\Phi$ along the $\lambda$-axis we need to know coefficients of $\mu^2, \mu^3, \ldots$ in $A_{\lambda,\mu}(2\pi)$

$-\sum_{m=0}^{\infty} A_\lambda^m \mu^m$. We have already obtained $A_\lambda^0$, $A_\lambda^1$; calculation of $A_\lambda^2$ is strai-

ghtforward using equations (E) and yields

$$A_\lambda^2 = A_2^2 = K^{-1}\begin{pmatrix} P_2 & \frac{1}{2\alpha}(U_2 + V_2) \\ \frac{\alpha}{2}(U_2 - V_2) & P_2 \end{pmatrix}$$

where

$$P_2 = \frac{\pi S}{2\alpha}$$

$$U_2 = \frac{S}{2}\left(\frac{1}{\alpha^2} + \frac{3}{\alpha^2 - 1}\right)$$

$$V_2 = \frac{4s}{K} - \frac{\pi c}{\alpha}$$

with $K = (4\alpha^2 - 1)$, $c = \cos 2\pi\alpha$, $s = \sin 2\pi\alpha$.

*Proposition* 2—The map $\Phi : \mathbf{R}^2 \to \overset{\circ}{\gamma} S \cap D$ has

(i) a Fold singularity at $(\lambda, \mu) = (1, 0)$ (i. e. $n = 2$);

(ii) a Cusp singularity at $(\lambda, \mu) = \left(\frac{9}{4}, 0\right)$ (i. e. $n = 3$).

PROOF : (i) Since $A_{\lambda,\mu}(2\pi) = A_\lambda^0 + \mu A_\lambda^1 + \mu^2 A_\lambda^2 + O(\mu_3)$ we have up to order $\mu^3$ :

$$u = \frac{1}{4K}(\beta U - \gamma V)$$

$$v = \frac{1}{4K}(-\gamma U + \beta V)$$

where $\beta = \alpha + \alpha^{-1}$, $\gamma = \alpha - \alpha^{-1}$, and

$$U = -4s\mu + U_2\mu^2$$

$$V = 2Ks + V_2\mu^2.$$

With $\alpha = 1 + \zeta$ we have

$$U = -8\pi\zeta\mu + \frac{3\pi\mu^2}{2}$$

$$V = 12\pi\zeta + 32\pi\zeta^2 - \pi\mu^2 + O(3)$$

or

$$U/A = \zeta u + C\mu^2 + O(3)$$

$$V/B = \zeta + h(\zeta, \mu)$$

where $A = -8\pi$, $B = 12\pi$, $C = -3/16$ and $h(\zeta, \mu) = O(2)$. Now

$$\zeta\mu + C\mu^2 = C\left(\frac{1}{2C}\zeta + \mu\right)^2 - \frac{1}{4C}\zeta^2$$

so if $X = \zeta + h(\zeta, \mu)$ and $(U', V'') = (V/B, U/AC)$ then

$$U' = X$$

$$V'' = \left(\frac{1}{2C}X + \mu\right)^2 - \frac{1}{4C^2}X^2 + O(3).$$

Finally, taking

$$Y = \frac{1}{2C}(\zeta + h) + \mu$$

$$V' = V'' + \frac{1}{4C^2}(U')^2$$

we obtain

$$(U', V') = (X, Y^2) + O(3)$$

and so there is a Fold at (1, 0) as claimed, since $(\zeta, \mu) \to (X, Y)$ and $(u, v) \to (U, V) \to (U', V')$ are coordinate transformations nonsingular at the origin.

(ii) To determine the singularity of $\Phi$ at $\left(\frac{9}{4}, 0\right)$ let $\alpha = \frac{3}{2} + \eta$; then we find as above

$$U/E = \eta\mu + O(3)$$

$$V/F = \eta + k(\eta, \mu)$$

where $E = 8\pi$, $F = -32\pi$ and $k(\eta, \mu) = O(2)$. Thus if $X = \eta + k(\eta, \mu)$, $Y = \mu$ and $(U', V'') = (V/F, U/E)$ then

$$U' = X$$

$$V'' = XY + m(X, Y)$$

where $m(X, Y) = O(3)$. Now a linear transformation of $(X, Y)$ to $(X, Y - \rho X)$ for some constant $\rho$ can be chosen to remove the $XY^2$ term in $m(X, Y)$, and a transformation of the form $(U', V') = (U', V'' - \sigma(U')^2 - \tau U' V'')$ can then be chosen to remove the $X^3$ and $X^2 Y$ term in $V''$. Thus we arrive at

$$(U', V') = (X, XY + dY^3) + O(4)$$

which proves that we have a Cusp, provided that $d \neq 0$.

The fact that $d$ is nonzero can be verified from explicit calculation of $A_\lambda^3$, requiring rather tedious calculations (compare Afsharnejad[1]). However, the result follows by a shorter but indirect argument as follows.

Since

$$u^2 - v^2 = \frac{1}{4K^2}(U^2 - V^2) = p^2 - 1 = (p - 1)(p + 1)$$

we see that $p = -1$ precisely when $U = \pm V$, i. e.

$$G(XY + m(X, Y)) = \pm X$$

where $G = EF^{-1}$. Now $m(x, Y)$ can be written in the form

$$m(X, Y) = m(O, Y) + Xr(X, Y)$$

and so $p = 1$ when $(X, Y)$ lies on one or other of the two curves

$$Gm(0, Y) = X(\pm 1 - GY - Gr(X, Y)),$$

i. e.

$$X = (\pm 1 + GY + O(2))\, Gm(O, Y).$$

By Theorem 1 (ii) we know that these two curves (the transition curves through $(\lambda, \mu) = \left(\frac{9}{4}, 0\right)$) have contact of order 3 at $(X, Y) = (0, 0)$, and so $Gm(0, Y)$ must have nonzero coefficient of $Y^3$, which is easily checked to be equivalent to the statment $d \neq 0$.

*Corollary*—The function $f = \pi \circ \Phi : \mathbf{R}^2 \to \mathbf{R}$ can be expressed in suitable local coordinates around $(\lambda_2, 0) = (1, 0)$ as

(i) $(x, y) \to 1 - x^2 + y^4$

and around $(\lambda_3, 0) = \left(\frac{9}{4}, 0\right)$ as

(ii) $(x, y) \to -1 + x^2 - y^6$.

PROOF : From $p^2 = 1 + u^2 - v^2$ we see that

$$f(\zeta, \mu) = \left(1 + \frac{1}{4K}(U^2 - V^2)\right)^{1/2}$$

where $U, V$ are functions of $(\zeta, \mu)$ as above; note that $K = 3 + 8\zeta + 4\zeta^2$.

Now

$$U^2 - V^2 = A^2 C^2 (V^4) - B^2 (U')^2$$

$$= A^2 C^2 \left(Y^2 - \frac{1}{4C^2} X^2 + O(3)\right)^2 - B^2 X^2$$

and so by replacing $X$ by $\bar{X} = X + H(X, Y)$ for a suitable function $H(X, Y)$ which is $O(2)$ we can complete the square to obtain

$$p = 1 + \frac{1}{8K^2}(A^2 C^2 Y^4 - B^2 \bar{X}^2) + O(5).$$

Then putting $2\sqrt{2K}(x, y) = (B\bar{X}, ACY)$ we obtain the stated result (i) up to terms of degree 4; finally, theory of determinacy of jets (i. e. truncated Taylor series) allows us to transform away the $O(5)$ terms : see e. g. references Brocker and Lander[7], Gibson[10], Poston and Stewart[25], Zeeman[27] for details.

The result (ii) follows by similar elementary manipulations, using the fact that $m(0, Y)$ has nonzero coefficient of $Y^3$, together with determinancy theory to remove $O(7)$ terms.

*Remark* : In case (i) the ratio of coefficients $A^2 C^2 : B^2$ is $1 : 64$, and the line $x = 0$ (tangent to the two transition curves $f(x, y) = 0$) corresponds to $V = 0$, i. e.

$$\zeta = \frac{1}{12}\mu^2 + O(\mu^3)$$

which agree with the calculations in Afsharnejad[1] where $A^3_\lambda$ , $A^4_\lambda$ were used to compute $f$ explicitly up to terms of degree 4.

Clearly, calculations to exibit the singularity of $\Phi$ at $(\lambda_{n,0})$ for $n \geqslant 4$ are going to be increasingly laborious. However, since the transition curves $\gamma'_n$ , $\gamma''_n$ through $(\lambda_n, 0)$ are known to have contact of order $n$, and the coefficient of $\zeta^2$ in $f(\lambda, \mu) = f(\alpha_n + \zeta)^2, \mu)$ is non zero (see $A^0_\lambda$ ) it is not too hard to deduce that in suitable coordinates around $(\lambda_n, 0)$ the function $f$ should have the form

$$f(x, y) = (-1)^n (1 - x^2 + y^n).$$

From this we can also derive

*Proposition* 3—The map $\Phi : \mathbf{R}^2 \to S \cap D$ has a singularity at $(\lambda, \mu) = (\lambda_n, 0)$ which in suitable coordinates has the form

$$\Phi(x, y) = (x, g(x, y))$$

where $g(0, y) = \pm y^n$.

It would be desirable to have an easy direct proof of this fact, not only because of the geometrical insight which it gives, but because it would lead to a proof that the $\frac{1}{2}$-discriminant function and the transition curves $\gamma'_n$ , $\gamma''_n$ do indeed have the form described above close to $(\lambda_n, 0)$.

5.2. *Perurbations of* $\Phi$ *at* $(\lambda_n, 0)$, $n \geqslant 2$

Since the map $\Phi : \mathbf{R}^2 \to S \cap D$ has a fold at (1,0) and is therefore (by Theorem 4) stable there, it follows that any sufficiently small perturbation of $\Phi$ can be brought

about by changing coordinates locally in $(\lambda, \mu)$-space $\mathbf{R}^2$ and in $S \cap D$ described locally by $(u, v)$ coordinates: the coordinate changes may involve shifts of the origin, however. Since the geometrical structure of $\Phi$ near (1, 0) is roughly as indicated in Fig. 8 it is intuitively easy to see that any perturbations $\Psi$ of $\Phi$ that intersect $S_3 \cup S_4$ in any way differently from $\Phi$ will have to do so in one or other of the ways indicated in Fig. 9 (i), (ii); then $\Psi^{-1}$ $(S_3 \cup S_4)$ must look as shown. Thus not all the changes in

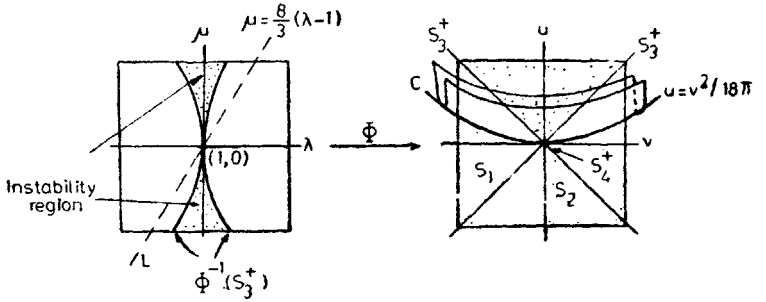


FIG. 8. Local structure of $\Phi$ near (1, 0). The fold occurs approximately along the line $L$ which is mapped by $\Phi$ approximately to the curve $C$.

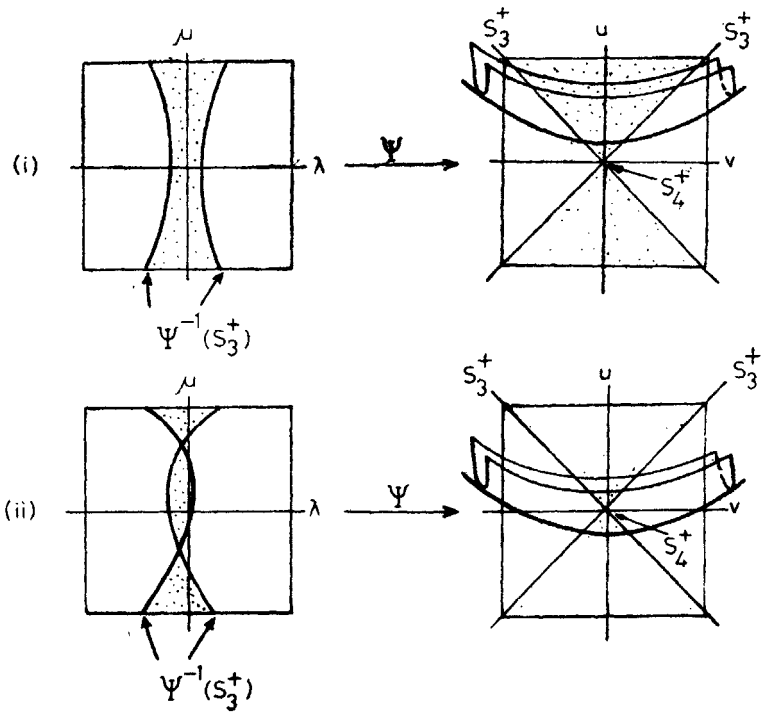


FIG. 9. Changes in the structure of the stability region when $\Phi$ is perturbed to $\Psi$ near $(\lambda, \mu) = (1, 0)$.

the zero set of $f \pm 1$ that are obtained by perturbing $f$ (and classified by the swallowtail, Fig. 6) can be realized by perturbing $\Phi$: in fact we obtain only those corresponding to the curve $\sigma$ of self-intersections of the swallowtail, and its continuation which is the "whisker" representing coincident complex critical points: see Fig. 6. This curve $\sigma$ corresponds to those functions $F$ whose $y$ terms are perfect square.

At the point $(\lambda_3, 0) = \left(\frac{9}{4}, 0\right)$ similar pictroial arguments using the stability of the cusp map show that not all perturbations of $f$ can be obtained by perturbing $\Phi: \mathbf{R}^2 \to S \cap D$, and the typically perturbed transition curves $\Psi^{-1}(S_3 \cup S_4)$ must (if qualitatively different from those of $\Phi$) apparently take one of the forms indicated in Fig. 10. We shall not pursue a more rigorous argument here, but merely observe that with the help of Proposition 3 it should not be difficult to prove this and analogous results for $(\lambda_n, 0)$, $n \geqslant 4$ by means of a suitable adaptation of the theory of versal deformations.

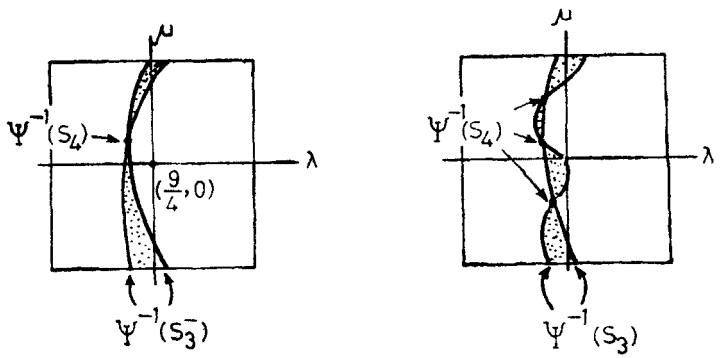


FIG. 10. Changes in the structure of the stability region when $\Phi$ is perturbed to $\Psi$ near (9/4,0).

In the above, all perturbations $\Psi$ of $\Phi$ have been assumed to satisfy $\Psi(\mathbf{R}^2) \subset D \cap S$ (which would be the case if the original Mathieu equation $(M)$ were perturbed to a Hill's equation $(H)$ with $Q(t)$ an even function).

We end by remarking on perturbations of $\Phi$ that are not constrained to satisfy $\Psi(\mathbf{R}^2) \subset D \cap S$. These arise, for instance, if in $(M)$ the term $\cos t$ is replaced by $\cos t + \epsilon q(t)$ where the $2\pi$ periodic $q(t)$ is not an even function (then we still have $\Psi(\mathbf{R}^2) \subset S$) or if a damping term is introduced (then $\Psi(\mathbf{R}^2) \not\subset S$). In this context $\Phi$ is not stable at $(\lambda_n, 0)$ for any $n \geqslant 1$, because as a map $\Phi: \mathbf{R}^2 \to \mathbf{R}^3$ or $\mathbf{R}^4$ (locally) it is highly degenerate. Nevertheless, on geometric grounds it seems plausible that 'typical' perturbations $\Psi$ met in practice should be such that $\Psi(\mathbf{R}^2) \cup (S^3 \cup S^4)$ is fairly simple. For example, by merely displacing $\Phi$ to $\Psi$ with $\Psi(\mathbf{R}^2) \cap S^4 = \phi$ locally

near $(\lambda, \mu) = (1, 0)$ we obtain all three types of perturbed transition curves as depicted in Fig. 6. Of course a typical perturbation of $\Phi : \mathbf{R}^2 \to \mathbf{R}^3, \mathbf{R}^4$ will not be just a displacement, but will open out to map $\mathbf{R}^2$ to a smooth surface in $\mathbf{R}^3$ or $\mathbf{R}^4$, possibly with some Umbrella points (Theorem 4) in the $\mathbf{R}^3$ case.

Note that the addition of a linear damping term does in fact change the transition curves near each $(\lambda_n, 0)$, $n \geq 1$ so that they take the form shown in Fig. 6 (region II) see Arnol'd[3] Nayfeh and Mook[24] In general the question of relating perturbations of $\Phi$ to perturbations of the original differential equation is an important and interesting one, needed to complete the picture in this geometric approach to the Mathieu equation. Some results are in Afsharnejad[1], but we leave a closer study to a subsequent paper currently in preparation.

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