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ON DISTRIBUTIVELY GENERATED NEAR-RINGS WITH GENERATORS FORMING INVERSE SEMIGROUPS

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In this paper equivalent conditions have been obtained for a distributively generated (d. g.) near-ring to have the generators forming an inverse semigroup with idempotent generators lying in the center. Conditions for the generators to form a group are also obtained. Conditions for a torsion free d. g. near-ring to have the generators forming either an inverse semigroup or a group are obtained.

1. DEFINITION AND NOTATION

Let $(S, .)$ be a semigroup. S is said to be regular if given $a \in S$, there is an $x \in S$ such that $axa = a$. Then ax and xa are idempotents. S is said to be an inverse semigroup if $(S, .)$ is regular in which all idempotents commute. Equivalently by Howie[1], a semigroup $(S, .)$ is an inverse semigroup if given $a \in S$, there exists a unique $b \in S$ such that $aba = a$ and $bab = b$. A near-ring N is a system $(N, +, .)$ such that $(N, +)$ is a group, $(N, .)$ is a semigroup and $(a + b)c = ac + bc$ for all a, b, c in N. N is zerosymmetric if $a0 = 0$ for all $a \in N$. An element $a \in N$ is said to be distributive if $a(b + c) = ab + ac$ for all b, c in N. A near-ring N is said to be regular if $(N, .)$ is regular. Given $x \in N$, if there exists an element $y \in N$ such that $xyx = x$ and $yxy = y$, then y is said to be inverse of x. A near-ring N is said to be a d.g. near-ring if there is a multiplicative subsemigroup D of distributive elements generating $(N, +)$. We write this d.g. near-ring as (N, D). For the basic terminology and notation we refer to Pilz[4]. The link between inverse semigroups and near-rings was first made by Mahmood *et al.*[3].

2. GENERATORS OF (N, D) FORMING INVERSE SEMIGROUP

Theorem 2.1—Let (N, D) be a d.g. near-ring in which $(D, .)$ is regular. Then, the following are equivalent :

(i) $(D, .)$ is an inverse semigroup with inverse of any element of D lying only in D.

(ii) For each $a \in D$, there is an $x \in D$ with $a = axa$, such that both ax and xa are central idempotents.

(iii) For any $a \in D$, if $a = aya$ for some $y \in N$, then $ay = ya$.

(iv) For any $a \in D$, if $a = aya = aza$ for some $y, z \in N$, then $ay = az = za = ya$.

PROOF : (i) implies (ii) : First we observe that for any $a \in D$ if there exists $x \in N$ such that $axa = a$, then xax is an inverse of a and hence is in D. Let $a \in D$ and let b be the unique inverse of a which lies in the inverse semigroup $(D, .)$. For any $y \in N$, we have $a(b + y - bay)\, a = a$. Hence $(b + y - bay)\, a\, (b + y - bay)$ is an inverse of a and it lies in D. So $b = (b + y - bay)\, a\, (b + y - bay) = (b + y - bay)\, ab = b + yab - bay\, ab$. Hence $yab = bay\, ab$. Since $a(b + y - yab)\, a = a$, we have as earlier $bay = bay\, ab$. So $bay = yab$. Letting $y = ab$ and $y = ba$ successively in the last equation we get $ab = ba^2b = ba$.

(ii) implies (iii) : Let $a \in D$. Choose an $x \in D$ with $axa = a$ such that ax and xa are central. Then $ax = (axa)\, x = a\,(xa)\, x = (xa)\,(ax) = x\,(ax)\, a = xa$. So if $a = aya$ for some $y \in N$, then $ay = (ax)\, ay = ayax = ax = xa = (xa)\, ya = yaxa = ya$.

(iii) implies (iv) : Let $a \in D$. Suppose $a = aya = aza$ for so me $y, z \in N$, then using (iii), $ay = ya = y\,(aza) = (ya)\,(za) = ayaz = az = za$.

(iv) implies (i) : Let $a \in D$. Since $(D, .)$ is regular, there exists an $x \in D$ such that $axa = a$. Let $b = xax$. Then b is an inverse of a which lies in D. Suppose $y, z \in N$ are inverses of a, then using (iv), $y = y\,(ay) = y\,(az) = (ya)\, z = zaz = z$. So (i) follows.

Corollary 2.2—Let (N, D) be a d.g. near-ring with $(D, .)$ a regular semigroup. The idempotents in D are central if and only if condition (i) of the above theorem is satisfied.

PROOF : If the idempotents in D are central, then condition (ii) of Theorem 2.1 is satisfied and hence condition (i) is true. Conversely if e is an idempotent in D, by the condition (ii) of Theorem 2.1, there exists an $x \in D$ such that $exe = e$ and ex is a central idempotent. Since $eee = e$, we have $ex = ee = e$ and hence e is in the center.

Corollary 2.3—If (N, D) is a d.g. near-ring with D an inverse semigroup such that inverse of any element of D lies only in D, then $(0 : e) = \{x \in N : xe = 0\}$ is an ideal where e is an idempotent in D.

PROOF : Clearly $(0 : e)$ is a left ideal. Since e lies in the centre, for any $n \in N$ and $x \in (0 : e)$, we have $xne = xen = 0n = 0$. So $xn \in (0 : e)$ and $(0 : e)$ is an ideal.

Now we give an example of a d.g. near-ring (N, D) which is not a ring with D an inverse semigroup and idempotents of D lying in the centre. This is the unique smallest d.g. near-ring with the above properties.

Example 1—On the dihedral group $\{0, a, 2a, 3a, b, a + b, 2a + b, 3a + b\}$ define multiplication as follows :

.	0	a	$2a$	$3a$	b	$a+b$	$2a+b$	$3a+b$
0	0	0	0	0	0	0	0	0
a	0	a	$2a$	$3a$	b	$a+b$	$2a+b$	$3a+b$
$2a$	0	$2a$	0	$2a$	0	0	0	0
$3a$	0	$3a$	$2a$	a	b	$a+b$	$2a+b$	$3a+b$
b	0	b	$2a$	$2a+b$	b	0	$2a+b$	$2a$
$a+b$	0	$a+b$	0	$a+b$	0	$a+b$	0	$a+b$
$2a+b$	0	$2a+b$	$2a$	b	b	0	$2a+b$	$2a$
$3a+b$	0	$3a+b$	0	$3a+b$	0	$a+b$	0	$a+b$

This example is the scheme (7) in Pilz[4] (p. 345). This is a d.g. near-ring with $D = \{a, a+b\}$ which is an inverse semigroup with idempotents of D lying in the center. The following example shows that there do exist d.g. near-rings (N, D) with D an inverse semigroup in which idempotents of D are not central.

Example 2—Consider the set N of Matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a,b,c,d lie in the field $GF[2]$.

If $D = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$, then (N, D) is a d.g. near-ring under matrix addition and matrix multiplication with $(D, .)$ as an inverse semigroup. Clearly the idempotent $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not central. This happens because $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ which is an inverse of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ lies outside D.

Mahmood *et al.*[3] have shown that if (N, D) is any d.g. near-ring with D an inverse semigroup, then $I = \{x \in (N, D) : ex \neq 0 \text{ for some idempotent } e \in D\}$ is an ideal. Now if D contains 0, then I will be N. But if we take $I' = \{x \in (N, D) : ex = 0 \text{ for some non-zero idempotent } e \in D\}$, then I' will not be an ideal as in the case of d.g. near-ring $N \equiv Z_6 = \{0, 1, 2, 3, 4, 5\}$ with $D = \{0, 3, 4\}$. The following Theorem finds the condition under which a d.g. near-ring (N, D) will have a proper ideal.

Theorem 2.4—If (N, D) is a d.g. near-ring with $(D, .)$ an inverse semigroup with inverse of any element of D lies only in D and if D does not contain identity, then N has a proper ideal.

PROOF : Suppose (N, D) has no proper ideal. For any non-zero idempotent $e \in D$, by corollary 2.3, $(0 : e)$ is an ideal. Since $e \notin (0 : e)$ we have $(0 : e) = 0$ and hence non-zero idempotents in D are not zero divisors. Now for any $a \in D$ and for any non-zero idempotent $e \in D$, $(a - ae)\, e = 0$. Hence $a = ae$ and every non-zero idempotent in D is a right identity in D. Let e, f be any two non-zero

idempotents in D, then $e = ef = fe = f$. Hence D contains a unique non-zero idempotent which is also the right identity, say e. For any $a \in D$, there is an $x \in D$ such that $a = axa$. Since ax is idempotent $a = ea$. So e is the identity for D, a contradiction.

Theorem 2.5—Let (N, D) be a d.g. near-ring in which $(D, .)$ is regular semigroup. For any $a \in D$, the following are equivalent :

(i) The solution for $a = axa$ is unique.

(ii) a is not a zero divisor.

(iii) N contains identity which lies in D and a is a unit.

PROOF : (i) implies (ii) : Let x be the unique solution of $a = axa$. Suppose $y \in N$ with either $ay = 0$ or $ya = 0$. Then $a(x + y)a = a$. Hence by (i) $x + y = x$. So $y = 0$.

(ii) implies (iii) : Suppose a is not a zero divisor. Let $x \in D$ satisfy $axa = a$. For any $y \in N$, we have $(y - yax)a = 0$. So $y = yax$. Since $a(y - xay) = 0$, we have $y = xay$. Hence ax is a right identity in N and xa is a left identity in N and they lie in D. Hence $ax = xa$ is the identity for N, which lies in D and clearly a is a unit.

(iii) implies (i) : Let x be the inverse of a. Clearly $axa = a$. Suppose there exists $y \in N$ such that $aya = a$, then $x = xax = x(aya)x = (xa)y(ax) = y$.

Remark 2.6 : Let N be any near-ring. If $a (\neq 0)$ is a distributive element such that $axa = a$ has a unique solution, then N contains identity.

Corollary 2.7—Let (N, D) be a d.g. near-ring in which $(D, .)$ is a regular semigroup. $(D, .)$ is a group if and only if for every $a \in D$, the solution for $axa = a$ is uniqne or a is not a zero divisor.

Corollary 2.8 (Ligh[2], Theorem 3.1)—Let N be a d.g. near-ring with more than one element. Then N is a division ring if and only if for each $a (\neq 0)$ in N there is a unique b in N such that $aba = a$.

PROOF : Since the only if part is immediate, we prove the other part. By remark 2.6, N contains identity, say 1. For any $a (\neq 0)$ in N, there exists a unique $b \in N$ such that $aba = a$. Since $ab(1)ab = ab$ and $ab(ab)ab = ab$ by uniqueness $ab = 1$. Similarly $ba = 1$. So b is the inverse of a and N becomes a near-field. Hence $(N, +)$ is abelian and N is a division ring.

3. TORSION FREE DISTRIBUTIVELY GENERATED NEAR-RING

A near-ring N is said to be torsion free if $(N, +)$ is torsion free.

Theorem 3.1—If (N, D) is a torsion free d.g. near-ring with $(D, .)$ a regular semigroup, then

(i) For any $a \in D$, if there exists finitely many $x \in N$ satisfying $axa = a$, then N has identity and $(D, .)$ is a group.

(ii) For any $a \in D$, if there exists a finitely many $x \in N$ satisfying $axa = a$ and $xax = x$, and if the idempotents of N lie in D, then $(D, .)$ is an inverse semigroup with idempotents lying in the center.

PROOF : (i) Let a be any element in D and let $A = \{x \in N : axa = a\}$. Clearly A is not empty. Suppose y is an element of N for which $ay = 0$ or $ya = 0$. Let $B = \{x + y : x \in A\}$. Then the map $\phi : A \to B$ given by $\phi(x) = x + y$ is an one-to-one and onto map. Since $B \subset A$, and A is finite, we have $A = B$. So if A contains the elements $\{x_1,...,x_k\}$ there is a permutation θ of indices $(1,...,k)$ such that $x_i = x_{\theta(i)} + y$ for $1 \leqslant i \leqslant k$. If m is the length of the orbit 1 under θ, then $x_1 = x_1 + my$. Hence $my = 0$. Since $m \geqslant 1$, we have $y = 0$. Hence by Theorem 2.5, a is invertible. Hence $(D, .)$ is a group.

(ii) Suppose (ii) is not true. By Theorem 2.1, we can find atleast one $a \in D$ such that $a = axa$ with $ax \neq xa$ for some $x \in N$. Let $xax = b$. Then $aba = a$, $bab = b$ with $ab \neq ba$. Now atleast one of the elements $ab - ba^2b$ and $ba - ba^2b$ is not zero. For otherwise $ab = ba$. Let $ab - ba^2b \neq 0$. Now for any integer $k = 1,2,...$, we have $a[b + k(ab - ba^2b)]a = a$ and $[b + k(ab - ba^2b)]a[b + k(ab - ba^2b)] = b + k(ab - ba^2b)$. Hence a has infinitely many inverses, a contradiction. Again, suppose $ba - ba^2b \neq 0$, we can show $[b + k(ba - ba^2b)]$ is an inverse of a. Hence we get an infinite number of $n \in N$ satisfying $ana = a$ and $nan = n$, a contradiction. Hence the theorem.

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