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ON TOPOLOGICAL PROJECTIVE PLANES-II

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In this paper we obtain a non-topological condition for the space of points of a topological projective plane to be homeomorphic to the space of lines and deduce thatin a topological Pappian plane the space of points is homeomorphic to the space of lines.

A topological projective plane is a projective plane in which the set of points $\mathscr{P}$ and the set of lines $\mathscr{L}$ are topological spaces and the operations of joining and intersecting are continuous in both variables. No known topological projective planes have non-homeomorphic topologies on the set of points and lines. In this paper we obtain a non-topological condition, in terms of the existence of certain projectivities, for the space of points to be homeomorphic to the space of lines. We also deduce that in a topological Pappian plane the space of points is homeomorphic to the space of lines. We assume the coordinatization of a Pappian plane by a field. We also use the following results :

1. Any projectivity from the points of one line to another is a homeomorphism. (Pickert[2], p. 262).
2. In a topological projective plane each ray of points is a closed set. (Pickert[2], p. 263).
3. A function f from a topological space X to a topological space Y is continuous iff for each net x_α in X which converges to a point x, the net $f(x_\alpha)$ converges to $f(x)$. (Kelley[1], p. 86).

We denote points by upper case latin letters $A, B, C, \ldots$ and lines by lower case latin letters $a, b, c, \ldots$. The line joining two distinct points A and B is denoted by AB.

Theorem 1—Let π be a topological projective plane. Let A, B, C be three non-collinear points and let $a = BC$, $b = CA$ and $c = AB$. Suppose there exist projectivities $f : a \to a$, g; $b \to b$ and $h : c \to c$ satisfying the following conditions :

(i) $f(B) = C$, $f(C) = B$,

$g(C) = A$, $g(A) = C$ and

$h(A) = B$, $h(B) = A$.

(ii) For any point $P \neq A, B, C$ the points $f(P A \cap a)$, $g(P B \cap b)$ and $h(P C \cap c)$ are collinear.

Then the space of points is homeomorphic to the space of lines.

PROOF : It follows easily from (i) and (ii) that for any line $l \neq a, b, c$ the lines $f(l \cap a) A$, $g(l \cap b) B$ and $h(l \cap c) C$ are concurrent. We now define $\sigma : \mathscr{P} \to \mathscr{L}$ as follows :

$$\sigma(P) = \begin{cases} f(P)\, A \text{ if } P \in a \\ g(P)\, B \text{ if } P \in b \\ h(P)\, C \text{ if } P \in c \\ \text{The line joining the three collinear points } f(P A \cap a), \\ g(P B \cap b) \text{ and } h(P C \cap c) \text{ otherwise.} \end{cases}$$

σ is a well defined bijection and its inverse is the function $\mu : \mathscr{L} \to \mathscr{P}$ defined by

$$\mu(l) = \begin{cases} f(l \cap a) \text{ if } A \in l \\ g(l \cap b) \text{ if } B \in l \\ h(l \cap c) \text{ if } C \in l \\ \text{The point of intersection of the three concurrent lines} \\ f(l \cap a) A,\ g(l \cap b) B \text{ and } h(l \cap c) C \text{ otherwise.} \end{cases}$$

We now prove that σ is continuous.

Let P_α be a net converging to P in $\mathscr{P}$.

Case 1 : $P \notin a, b, c$—Since the set of all points on any line is closed we may assume $P_\alpha \notin a, b, c$ for all α. It follows from the continuity of joining, intersection and the projectivities f, g and h that $\sigma(P_\alpha)$ converges to $\sigma(P)$.

Case 2 : $P \in a$ and $P \neq B, C$. If $P_\alpha \in a$ for all α, it follows from the continuity of joining and f that $\sigma(P_\alpha)$ converges to $\sigma(P)$.

Suppose $P_\alpha \notin a$ for all α.

Then $f(P_\alpha A \cap a) \to f(P A \cap a) = f(P)$ and $g(P_\alpha B \cap b) \to g(P B \cap b) = g(C) = A$.

Hence $\sigma(P_\alpha) \to f(P)\, A = \sigma(P)$.

The remaining cases can be similarly dealt with. Thus if P_α converges to P, then $\sigma(P_\alpha)$ converges to $\sigma(P)$ and hence σ is continuous. By a dual argument we can prove the continuity of the inverse function μ and hence σ is a homeomorphism.

Theorem 2—In a topological Pappian plane π the space of points is homeomorphic to the space of lines.

PROOF : Let F be the ternary ring of π. Let $O\ E\ U\ V$ be the quadrangle of reference. Let $O = (0, 0)$, $U = (0)$ and $V = (\infty)$

f: $O\ U \to O\ U$ defined by

$f((x, 0)) = (x^{-1}, 0)$ if $x \neq 0$,

$f((0, 0)) = (0)$ and

$f((0)) = (0, 0)$

is a projectivity mapping O to U and U to O.

g : $U\ V \to U\ V$ defined by

$g((m)) = (m^{-1})$ if $m \neq 0$,

$g((0)) = (\infty)$ and

$g((\infty)) = (0)$

is a projectivity mapping U to V and V to U.

h : $O\ V \to O\ V$ defined by

$h((0, y)) = (0, -y^{-1})$ if $y \neq 0$

$h((0, 0)) = (\infty)$ and

$h((\infty)) = (0, 0)$

is a projectivity mapping O to V and V to O. Now for any point $P = (x, y)$ where $x \neq 0 \neq y$, $f(V\ P \cap O\ U) = (x^{-1}, 0)$, $g(O\ P \cap U\ V) = (xy^{-1})$ and $h(U\ P \cap O\ V) = (0, -y^{-1})$ and these points are collinear. Hence by Theorem 1, the space of points is homeomorphic to the space of lines.

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