POLYNOMIAL RINGS AND MULTIPLICATION RINGS

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Andresen [Can. J. Math. 28 (1976), 760-68] has given a necessary and sufficient condition for a ring $R$ such that $R[x]$ is (1) a multiplication ring (2) an almost multiplication ring. We derive here a necessary and sufficient condition for a ring $R$ so that $R[x]$ is a ring with ($\ast$)-condition.

Throughout, $R$ denotes a commutative ring with unity. $R$ is called a multiplication ring if for its any ideals $A$ and $B$ with $A \subseteq B$, there is an ideal $I$ such that $A = IB$; equivalently, for any prime ideal $P$ and any ideal $A \subseteq P$, $A = IP$ for some ideal $I$ of $R$ (Mott, Theorem 9.21). $R$ is called an almost multiplication ring ($AM$-ring) if for each prime ideal $P$ of $R$, $R_P$ is a multiplication ring; equivalently, every ideal with prime radical is prime power [Butts and Phillips, Theorem 2.7]. $R$ is said to satisfy ($\ast$)-condition if every ideal with prime radical is primary in $R$. It is well known that an $AM$-ring always satisfies ($\ast$)-condition [Gilmer and Mott, Theorem 4; and Butts and Phillips, Theorem 2.9].

Anderson has proved that $R[x]$ is a multiplication ring if and only if $R$ is a finite direct product of fields and $R[x]$ is an $AM$-ring if and only if $R$ is a von Neumann ring. We prove here that $R[x]$ satisfies ($\ast$)-condition if and only if $R$ is a von Neumann ring, equivalently $R[x]$ is an $AM$-ring.

The following Lemma is due to Gilmer.

Lemma 1—If $R$ satisfies ($\ast$)-condition and $P, P'$ are prime ideals of $R$ with $P \subseteq P'$ then for any $p \in P$, there exists $p' \in P'$ such that $p = pp'$.

Proof: See Theorem 7 of Gilmer.

Lemma 2—If $R[x]$ satisfies ($\ast$)-condition then for any prime ideal $P$ of $R$, $R_P$ is a field.

Proof: Since $R/P$ is isomorphic to $R[x]/(P[x] + (x))$, it follows that $P[x] + (x)$ is a prime ideal of $R[x]$. Let $p \in P$ be any element. By Lemma 1, there exist $p(x) \in P[x]$ and $f(x) \in R[x]$ such that $p = p(p(x) + xf(x))$ as $P[x] \subseteq P[x] + (x)$ are prime ideals in $R[x]$. Hence $p(1 - p_0) = 0$ for some $p_0 \in P$. Then $1 - p_0 \notin P$ and if $\phi$ denotes the natural mapping from $R$ to $R_P$ then $\phi(1 - p_0)$ is a unit in $R_P$, forcing $\phi(p) = 0$. It follows that $P R_P = (0)$. $P R_P$ being the maximal ideal of $R_P$, $R_P$ is a field.
Theorem: $3 - R[x]$ satisfies (*)-condition if and only if $R$ is a von Neumann ring.

Proof: If $R$ is von Neumann then $R[x]$ is an AM-ring and hence satisfies (*)-condition. Conversely if $R[x]$ satisfies (*)-condition then $R_P$ is a field for every prime ideal $P$. Thus every $R$-module is flat [Atiyah and Macdonald, (Prop. 3.10)]. It follows that $R$ is a von Neumann ring.

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References