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POLYNOMIAL RINGS AND MULTIPLICATION RINGS

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Andesron [*Can. J. Math.* 28 (1976), 760-68)] has given a necessary and sufficient condition for a ring R such $R[x]$ is (1) a multiplication ring (2) an almost multiplication ring. We derive here a necessary and sufficient condition for a ring R so that $R[x]$ is a ring with (*)-condition.

Throughout, R denotes a commutative ring with unity. R is called a multiplication ring if for its any ideals A and B with $A \subseteq B$, there is an ideal I such that $A = IB$; equivalently, for any prime ideal P and any ideal $A \subseteq P$, $A = IP$ for some ideal I of R (Mott[9], Theorem 9.21). R is called an almost multiplication ring (AM-ring) if for each prime ideal P of R, R_P is a multiplication ring; equivalently, every ideal with prime radical is prime power [Butts and Phillips[3], Theorem 2.7]. R is said to satisfy (*)-condition if every ideal with prime radical is primary in R. It is well known that an AM-ring always satisfies (*)-condition [Gilmer and Mott[5] Theorem 4: and Butts and Phillips[3] Theorem 2.9].

Anderson[1] has proved that $R[x]$ is a multiplication ring if and only if R is a finite direct product of fields and $R[x]$ is an AM-ring if and only if R is a von Neumann ring. We prove here that $R[x]$ satisfies (*)-condition if and only if R is a von Neumann ring, equivalently $R[x]$ is an AM-ring.

The following Lemma is due to Gilmer[4].

Lemma 1—If R satisfies (*)-condition and P, P' are prime ideals of R with $P \subseteq P'$ then for any $p \in P$, there exists $p' \in P'$ such that $p = pp'$.

PROOF : See Theorem 7 of Gilmer[4].

Lemma 2—If $R[x]$ satisfies (*)-condition then for any prime ideal P of R, R_P is a field.

PROOF : Since R/P is isomorphic to $R[x]/(P[x] + (x))$, it follows that $P[x] + (x)$ is a prime ideal of $R[x]$. Let $p \in P$ be any element. By Lemma 1, there exist $p(x) \in P[x]$ and $f(x) \in R[x]$ such that $p = p(p(x) + xf(x))$ as $P[x] \subsetneq P[x] + (x)$ are prime ideals in $R[x]$. Hence $p(1 - p_0) = 0$ for some $p_0 \in P$. Then $1 - p_0 \notin P$ and if ϕ denotes the natural mapping from R to R_P then $\phi(1 - p_0)$ is a unit in R_P, forcing $\phi(p) = 0$. It follows that $PR_P = (0)$. PR_P being the maximal ideal of R_P, R_P is a field.

Theorem : 3— $R[x]$ satisfies (*)-condition if and only if R is a von Neumann ring.

PROOF : If R is von Neumann then $R[x]$ is an AM-ring and hence satisfies (*)-condition. Conversely if $R[x]$ satisfies (*)-condition then R_P is a field for every prime ideal P. Thus every R-module is flat [Atiyah and Macdonald[2], (Prop. 3.10)]. It follows that R is a von Neumann ring[7].

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