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A SUFFICIENT CONDITION FOR THE VALIDITY OF CHANDRASEKHAR'S CONJECTURE IN MAGNETOCONVECTION

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A simple mathematical proof is presented to establish a new result, namely, that at the marginal state in the magnetohydrodynamic simple Bénard problem the total kinetic energy associated with a disturbance is greater than or equal to its total magnetic energy, in the regime $Q\sigma_1 \leqslant 3$ where Q is the Chandrasekhar number and σ_1 is the magnetic Prandtl number. This result, which was long sought after, is applicable for quite general boundary conditions and provides a sufficient condition for the validity of Chandrasekhar's conjecture in magnetoconvection.

1. INTRODUCTION

Pellew and Southwell's[1] celebrated result establishes the validity of the principle of exchange of stabilities for the simple Bénard problem. No such result, however, did exist for a long time since then, in the magnetohydrodynamic case, when a uniform vertical magnetic field opposite to gravity is impressed upon the system and the possibility of the occurrence of oscillatory motions, in this case, has been a matter of speculation only in the literature on the subject. Chandrasekhar[2] extended the proof of the exchange principle developed by Pellew and Southwell[1] for this more general problem and concluded that a sufficient condition for the validity of the exchange principle is that the total kinetic energy associated with a disturbance is greater than or equal to its magnetic energy. Since, however, one cannot *a priori* be certain when this sufficient condition will be satisfied, Chandrasekhar[2] sought without success the regime in terms of the parameters of the system alone in which the validity of his sufficiency condition will be ensured and remarked"... it will appear that the method of Pellew and Southwell is not quite strong enough to establish the principle for the problem in hand". Unfortunately, he did not present a rigorous proof of his sufficiency condition and as a consequence it came to be known in the literature on magnetoconvection as Chandrasekhar's conjecture. A rigorous derivation of this conjecture together with the establishment of its regime of validity in the parameter space of the system

alone were much sought after questions to be resolved since Chandrasekhar's[2] work. Sherman and Ostrach[3] successfully answered the first question in a more general context when the fluid is completely confiined in an arbitrary region and the uniform magnetic field is applied in any arbitrary direction, but failed to resolve the second. Banerjee *et al.*[4] in a recent paper obtained a sufficiency condition in terms of the parameters of the system alone for the validity of the exchange principle by utilising a method which is essentially that of Pellew and Southwell[1] and this invalidiated Chandrasekhar's remark, but, surprisingly, they failed to see the connection between their own work and Chandrasekhar's conjecture.

An answer to the second question is given in the present paper. A simple mathematical proof is presented to establish that Chandrasekhar's conjecture is valid in the regime $Q\sigma_1 \leqslant 3$. This result is uniformly applicable for any combination of a dynamically free or rigid boundary when the regions outside the fluid are perfectly conducting or insulating.

2. Basic Equations and Boundary Conditions

The basic equations and boundary conditions in their nondimensional forms, for the magnetohydrodynamic simple Bénard problem when a uniform vertical magnetic field opposite to gravity is impressed upon the system are given by[4].

$$(D^3 - a^2)\left(D^2 - a^2 - \frac{p}{\sigma}\right) w = Ra^2\,\theta - Q\,D\,(D^2 - a^2)\,h_z \qquad \ldots(1)$$

$$(D^2 - a^2 - p)\,\theta = -\,w \qquad \ldots(2)$$

$$\left(D^2 - a^2 - \frac{p\sigma_1}{\sigma}\right) h_z = -\,D\,W \qquad \ldots(3)$$

$w = 0$ on both the boundaries ...(4)

$\theta = 0$ on both the boundaries ...(5)

$D^2 w = 0$ on a dynamically free boundary ...(6)

$Dw = 0$ on a rigid boundary ...(7)

$h_z = 0$ on both the boundaries if the regions outside the fluid are perfectly conducting ...(8)

$Dh_z = \mp a\,h_z$ on the boundaries if the regions outside the fluid are insulating ...(9)

where z is the vertical coordinate, $z = 0$ and $z = 1$ represent the two boundaries, $D \equiv \frac{d}{dz}$, w is the vertical velocity, θ is the temperature h_z is the vertical magnetic field, R is the Rayleigh number, a^2 is the square of the wave number, and $p = p_r + i\,p_i$, ir the complex growth rate , while the other symbols are as explained earlier. We note that R, Q, σ, σ_1, and a^2 are non-negative numbers.

Equations (1) — (3) and appropriate boundary conditions from (4) — (9) pose an eigenvalue problem for p and we wish to characterise p_I when $p_r \geqslant 0$.

3. Mathematical Analysis

We first treat the case when the regions outside the fluid are insulating and prove the following theorem :

Theorem 1—If $p_r \geqslant 0$ and $Q\,\sigma_1 \leqslant 3$ then eqn. (3) and boundary conditions (4) and (8) imply that

$$\int_0^1 (|Dw|^2 + a^2|w|^2)\,dz \geqslant Q\,\sigma_1 \int_0^1 (|Dh_z|^2 + a^2|h_z|^2)\,dz.$$

Proof : Multiplying (3) by h_z^* (an asterisk denotes complex conjugation) and integrating the resulting equation over the range of z, we have

$$\int_0^1 h_z^* \left(D^2 - a^2 - \frac{p\,\sigma_1}{\sigma}\right) h_z\,dz = -\int_0^1 h_z^*\,Dw\,dz \qquad \ldots(10)$$

Integrating (10) by parts for a suitable number of times and making use of the boundary conditions (9), we obtain from the real part of the resulting equation

$$a\{(|h_z|^2)_0 + (|h_z|^2)_1\} + \int_0^1 (|Dh_z|^2 + a^2|h_z|^2)\,dz$$

$$+\frac{p_r\,\sigma_1}{\sigma}\int_0^1 |h_z|^2\,dz$$

$$= \text{Real part of } \int_0^1 h_z^*\,Dw\,dz \qquad \ldots(11)$$

$$\leqslant \left|\int_0^1 h_z^*\,Dw\,dz\right|$$

$$\leqslant \int_0^1 |h_z^*\,Dw|\,dz$$

$$\leqslant \int_0^1 |h_z|\,|Dw|\,dz$$

(*equation continued on p.* 1322)

$$\leqslant \int_0^1 \frac{(|h_z|^2 + |Dw|^2)}{2}\, dz. \qquad \ldots(12)$$

Further, (11) and boundary conditions (4) also imply

$$a\{(|h_z|^2)_0 + (|h_z|^2)_1\} + \int_0^1 (|Dh_z|^2 + a^2|h_z|^2)\, dz$$

$$+ \frac{p_r \sigma_1}{\sigma} \int_0^1 |h_z|^2 dz$$

$$= -\text{ Real part of } \int_0^1 w\, Dh_z^*\, dz \qquad \ldots(13)$$

$$\leqslant \left| \int_0^1 w\, Dh_z^*\, dz \right| \leqslant \int_0^1 |w\, Dh_z^*|\, dz$$

$$\leqslant \int_0^1 |w|\,|Dh_z|\, dz$$

$$\leqslant \int_0^1 \frac{(|w|^2 + |Dh_z|^2)}{2}\, dz. \qquad \ldots(14)$$

Since $p_r \geqslant 0$, we derive from (12) and (14)

$$2\int_0^1 (|Dh_z|^2 + a^2|h_z|^2)\, dz \leqslant \int_0^1$$

$$\frac{(|h_z|^2 + |Dw|^2 + |w|^2 + |Dh_z|^2)}{2}\, dz \qquad \ldots(15)$$

and hence

$$\int_0^1 (|Dh_z|^2 + a^2|h_z|^2)\, dz \leqslant \tfrac{1}{3}\int_0^1 (|Dw|^2 + a^2|w|^2)\, dz. \qquad \ldots(16)$$

Therefore, if, $Q\sigma_1 \leqslant 3$, we obtain from (16)

$$\int_0^1 (|Dw|^2 + a^2|w|^2)\, dz \geqslant Q\sigma_1 \int_0^1 (|Dh_z|^2 + a^2|h_z|^2) dz \qquad \ldots(17)$$

and this establishes the theorem.

When the regions outside the fluid are perfectly conducting we have from (7)

$$a\{(|h_z|^2)_0 + (|h_z|^2)_1\} = 0 \qquad \ldots(18)$$

and it then follows from the above analysis that (17) is true in the regime $Q\sigma_1 \leqslant 3$. We thus have the following theorem :

Theorem 2—If $p_r > 0$ and $Q\sigma_1 \leqslant 3$ then equations (3) and boundary conditions (4) and (8) imply that

$$\int_0^1 (|Dw|^2 + a^2|w|^2)\,dz \geqslant Q\sigma_1 \int_0^1 (|Dh_z|^2 + a^2|h_z|^2)\,dz \ldots (19)$$

We note[3] that the left-hand sides of (17) or (19) represent the total kinetic energy associated with a disturbance while the right hand represent the total magnetic energy associated with a disturbance and thus we may state Theorem 1 and Theorem 2 in the following equivalent combined form :

Theorem 3—At the marginal or unstable state in the magnetohydrodynamic simple Bénard problem, for any combination of a dynamically free or a rigid boundary when the regions outside the fluid are perfectly conducting or insulating, the total kinetic energy associated with a disturbance is greater than or equal to its total magnetic energy in the regime $Q\sigma_1 \leqslant 3$.

This proves Chandrasekhar's conjecture in the region $Q\sigma_1 \leqslant 3$. Surprisingly, Banerjee *et al.*[4] did not pursue their investigation in this direction and as a consequence failed to see connection between their own work and Chandrasekhar's conjecture as mentioned earlier. The mathematical analysis presented in this paper, however, has surely an edge over that Banerjee *et al.*[4] for the reasons of, its striking simplicity and directness, showing that the validity of Chandrasekhar's conjecture is a consequence of only the equation of magnetic induction together with the magnetic boundary condition and the fact that the horizontal boundaries are fixed, and being the first in this context to explicitly claim a long sought after result.

References

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