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COUPLED SECOND ORDER BOUNDARY VALUE PROBLEMS

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A variational principle is derived for the numerical solution of coupled second order boundary value problems. Conventionally in the global variational approach for the solution of such problems, the trial functions are chosen to explicitly satisfy the essential boundary conditions associated with the differential equations. This constraint can, in many circumstances, prove inconvenient and it is preferable to work within a variational formalism which does not require this condition on the trial functions. In this paper, we give a uniform framework for treating coupled problems where the boundary conditions are incorporated via a functional which is stationary at the solution and hence the trial functions need not satisfy any of them.

1. INTRODUCTION

Variational principles derived so far for the solution of coupled second order boundary value problems, as described for instance by Smith[6,7], and Barrett[1], require the approximation functions to satisfy exactly the boundary conditions of the problem. This difficulty has been overcome in this paper by using a technique where the boundary conditions are imposed implicitly via suitable terms in the functional and hence the expansion functions are not required to satisfy any of them. Within the region R of the coupled problem :

$$-\nabla . A(\mathbf{X}) \nabla U_1(\mathbf{X}) + B_1(\mathbf{X}) U_2(\mathbf{X}) = D_1(\mathbf{X}), \mathbf{X} \in R \qquad ...(1.1)$$

$$-\nabla . A(\mathbf{X}) \nabla U_2(\mathbf{X}) + B_2(\mathbf{X}) U_1(\mathbf{X}) = D_2(\mathbf{X})$$

a global expansion of the solution is made :

$$U_k(\mathbf{X}) \approx U_N^{(k)}(\mathbf{X}) = \sum_{i=1}^{N} a_i^{(k)} h_i(\mathbf{X}); K = 1, 2, \mathbf{X} \in R \qquad ...(1.2)$$

where $h_i(\mathbf{X})$ is a set of suitable basis—typically, a set of orthogonal polynomials (see sections 2 and 3 for more details). Inserting these expansions into an appropriate functional and finding the stationary value of the functional leads to the defining equations for the coefficients $a_i^{(1)}$ and $a_i^{(2)}$.

In the next section we introduce a functional which is stationary at the solution of a given boundary value problem for a class of functions which do not satisfy any of the boundary conditions.

2. A Functional Embodying the Boundary Conditions

Our first result specifies a functional which is stationary at the solution of the following boundary value problem :

$$\nabla . A(\mathbf{X}) \nabla U_1(\mathbf{X}) + B_1(\mathbf{X}) U_2(\mathbf{X}) = D_1(\mathbf{X}), \mathbf{X} \in R \quad \ldots(2.1a)$$

$$\nabla . A(\mathbf{X}) \nabla U_2(\mathbf{X}) + B_2(\mathbf{X}) U_1(\mathbf{X}) = D_2(\mathbf{X})$$

Subject to the following boundary conditions, with boundary ∂R :

$$U_1(\mathbf{X}) = g_1(\mathbf{X}), \mathbf{X} \in \partial R \quad \ldots(2.1b)$$

$$U_2(\mathbf{X}) = g_2(\mathbf{X})$$

we introduce the following adjoint problem with solution $V_1(\mathbf{X})$ and $V_2(\mathbf{X})$:

$$\nabla . A^T(\mathbf{X}) \nabla V_1(\mathbf{X}) + B_1(\mathbf{X}) V_2(\mathbf{X}) = D_1(\mathbf{X}), \mathbf{X} \in R \quad \ldots(2.2a)$$

$$\nabla . A^T(\mathbf{X}) \nabla V_2(\mathbf{X}) + B_2(\mathbf{X}) V_1(\mathbf{X}) = D_2(\mathbf{X})$$

with boundary conditions

$$V_1(\mathbf{X}) = g_1(\mathbf{X}), \mathbf{X} \in \partial R \quad \ldots(2.2b)$$

$$V_2(\mathbf{X}) = g_2(\mathbf{X}).$$

Let $\mathbf{n}$ be the outward normal; ds the differential along the boundary ∂R with positive direction being counterclockwise.

Theorem—The solutions to problems (2.1) and (2.2) render the following functional stationary at the point $U_k = u_k$ and $V_k = v_k$; $k = 1, 2$:

$$\begin{aligned} F(u_1, u_2, v_1, v_2) = \int_R & [\nabla v_2 \cdot A \nabla u_1 + B_1 u_2 v_2 + B_2 u_1 v_1 \\ & - D_1(u_2 + v_2) - D_2(u_1 + v_1)] \, d\mathbf{X} \\ & + \int_{\partial R} [(g_2 - v_2)(A \nabla u_1) + (g_1 - v_1)(A \nabla u_2) \\ & + (g_2 - u_2)(A^T \nabla v_1) + (g_1 - u_1)(A^T \nabla v_2)]. \\ & \mathbf{n} \, ds. \end{aligned} \quad \ldots(2.3)$$

PROOF : Let $\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$ and $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ be the solution vectors of problems (2.1) and (2.2) respectively; W_1, W_2, W_3 and W_4 arbitrary functions.

Define

$$G_1(\epsilon) = F(U_1 + \epsilon W_1, U_2, V_1, V_2),$$

$$G_2(\epsilon) = F(U_1, U_2 + \epsilon W_2, V_1, V_2),$$

$$G_3(\epsilon) = F(U_1, U_2, V_1 + \epsilon W_3, V_2),$$

$$G_4(\epsilon) = F(U_1, U_2, V_1, V_2 + \epsilon W_4).$$

Then F is stationary at U_1, U_2, V_1 and V_2 if

$$\frac{dG_l(0)}{d\epsilon} = 0 \text{ for } l = 1, 2, 3, 4.$$

Now,

$$\frac{dG_1(0)}{d\epsilon} = \int_K [\nabla V_2 \, A \, \nabla W_1 + B_2 V_1 W_1 - D_2 W_1] \, d\mathbf{X} + \int_{\partial K} [(g_2 - V_2)(A \nabla W_1) - W_1 (A^T \nabla V_2)] . \mathbf{n} \, ds. \quad \ldots(2.4)$$

The first integral in (2.4) reduces by Green's theorem and (2.2) to

$$\int_R [\nabla W_1 . A^T \nabla V_2 + B_2 V_1 W_1 - D_2 W_1] \, d\mathbf{X} = \int_{\partial R} W_1 A^T \nabla V_2 . n \, ds$$

This line integral cancels the corresponding integral in (2.4) taking into account the boundary conditions of (2.2b). Hence $dG_1(0)/d\epsilon = 0$ and F is stationary at V_2. Similarly it can be proved that $dG_m(0)/d\epsilon = 0$, $m = 2, 3, 4$. Whence the functional F is stationary at the solutions of (2.1) and (2.2).

We introduce an appropriate expansion set and the solution W_1 and W_2 of (2.1) in approximated as :

$$U_k(\mathbf{X}) \sim U_k^{(N)} = \sum_{i=1}^{N} a_i^{(k)} h_i(\mathbf{X}), \; K = 1, 2. \quad \ldots(2.5)$$

For an error analysis of the truncation error due to the cutting off the expansion at the Nth term, see Delves[2] where a very similar treatment is given for integral equation problems. Inserting (2.5) into (2.3) and finding the stationary value of the functional leads to the defining equations for the vectors $\mathbf{a}^{(K)}$:

$$\begin{bmatrix} H & G^{(1)} \\ G^{(2)} & H \end{bmatrix} \begin{bmatrix} \mathbf{a}^{(1)} \\ \mathbf{a}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}^{(1)} \\ \mathbf{Q}^{(2)} \end{bmatrix} \quad \ldots(2.6)$$

Equation (2.6) is a 2×2 block-matrix equation where for $i, j = 1, 2, \ldots, N$ and $K = 1, 2$

$$H_{ij} = \int_R \nabla h_j . A \nabla h_i \, \mathrm{dx} - \int_{\partial R} (h_j A \nabla h_i + h_i . A^T \nabla h_j) . \mathbf{n} \, ds \qquad \ldots(2.7)$$

$$G_{ij}^{(k)} = \int_R h_i B_k h_j \, dx$$

$$Q_j^{(k)} = \int_R D_k h_j \, dx + \int_{\partial R} (g_k A^T \nabla h_j) . \mathbf{n} \, ds.$$

The matrices H, $G^{(k)}$, and the vectros $\mathbf{Q}^{(k)}$ of (2.7) are similar in structure to those obtained by Delves and Hall[3] when the Global clement formalism is applied to the solution of elliptic partial differential equations. Therefore, the computer package of Phillips *et al.*[5] which solves the block-matrix equation of Delves and Hall[3] can be used with minor changes to solve (2.6). Next, for two-dimensional problems we set

$$h_i (x, y) = k_p (x) \, l_q (y) \qquad \begin{matrix} i = 1. 2, \ldots, N \\ p = 1, 2, \ldots, N_p \\ q = 1, 2, \ldots, N_q \end{matrix} \qquad \ldots(2.8)$$

where $i \to (p, q)$ is a bijection and $N = N_p N_q$; N is the number of expansion functions and choose

$$K_p (X) = l_p (X) = h_p (X) \qquad \ldots(2.9)$$

(see next section for the choice of $h_p (X)$ and the reason for that choice)

3. Numerical Results

To test the formalism of this paper, we consider as an example the numerical solution of the following two dimensional coupled problem:

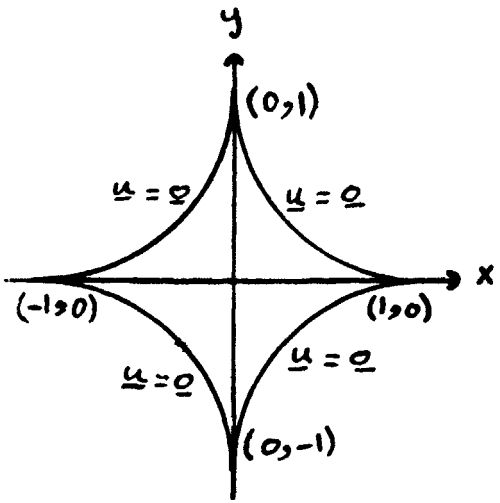
$$-\nabla^2 u_1 = 1, \; u_1 = 0 \text{ on the boundary} \qquad \ldots(3.1a)$$

$$-\nabla^2 u_2 + 100 u_1 = 0, \; u_2 = 0 \text{ on the boundary.} \qquad \ldots(3.1b)$$

Where the rcgion is cuspidal formed of four arcs of circles with radius 1 and origin at the centre of the region as shown in Figure 1. This problem is of interest here because it can be used as a test problem for the formalism of this pa per since it can be solved in two different ways as we shall show in what will follow. We choose the basis functions $h_i (X)$ in eqn. (2.9) as:

$$h_1 (X) = 1, \; h_2 (X) = X, \; h_i (X) = (1 - X^2) \, T_{i-3} (X); \; i = 3, 4, \ldots\ldots(3.2)$$

Where the $T_{i-3} (X)$ are chebyshev polynomials of the first kind. The reason for this choice is the need to handle the derivative terms in the matrix H in (2.6) without in-



$u = [u_1, u_2]^T$

FIG. 1.

troducing artificial singularties. With this choice of basis functions, calculations were carried out using the program package of Phillips *et al.*[5], after suitable modification and the results for $N = 5$ (the number of expansion functions) are given in Table I. Problem (3.1) is chosen in particular as a test problem because it can be solved by a different approch then the one given in this paper. It can be solved using the formalism of Delves and Hall[3] and the computational technique of Delves and Phillips[4] as two "single" elliptic boundary value problems in the following way :

TABLE I

Computed solutions for $N = 5$

X	y	u_1	u_2
0.0000	0.0000	5.3940, −2	−2.0556, − 1
0.3880	0.0000	2.3957, − 2	−7.8269. − 2
0.1670	0.1670	3.7803, − 2	− 1.3341, — 1
1.0000	0.0000	0.0000, 0	0.0000,0
0.6170	0.0761	0.0000, 0	0.0000,0
0.2930	0.2930	0.0000, 0	0.0000, 0
0.0000	1.0000	0.0000, 0	0.0000, 0

Equation (3.1a) is solved for u_1 and its current value at each mesh point is substituted in (3.1b) which is then solved to get the corresponding value of u_2 at that mesh

point. Using this approach and the expansion functions (2.8) and (2.9) together with the computer program of Phillips *et al.* [5],calculations were carried out and the results for $N = 5$ are given in Table II.

TABLE II[1]

Computed Solutions for $N = 5$

X	Y	u_1	u_2
0.0000	0.0000	5.3940, — 2	— 2.0556, — 1
0.3880	0.0000	2.3957, — 2	— 7.8269, — 2
0.1670	0.1670	3.7803, — 2	— 1.3341, — 1
1.0000	0.0000	1.2790, — 13	9.0949, — 13
0.6170	0.0761	1.3500, — 13	5.1159, — 13
0.2930	0.2930	1.8474, — 13	1.1369, — 12
0.000	1.0000	1.2790, — 13	9.0949, — 13

Because of symmetry (see Figure 1) we tabulate the solutions of u_1 and u_2 at mesh points in the first quadrant only.

The agreement between the results of Table I and those of Table II for $N = 5$ is excellent. This excellent agreement is born out in the calculations for various number of expansion functions of which those for $N = 5$ have been tabulated. From this model calculation we draw the conclusion that the variational principle derived in this paper indeed gives an attractive approach and yeilds a straight forward treatment of coupled-second order boundary value problems.

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