Indian J. pure appl. Math., 17 (11) : 1330–1339, November 1986

AXISYMMETRIC VIBRATIONS OF ANNULAR SANDWICH PLATES OF PARABOLICALLY VARYING THICKNESS

A. P. GUPTA AND MANJU JAIN

Department of Mathematics, University of Roorkee, Roorkee 247667

(*Received* 6 *May* 1985; *after revision* 10 *June* 1986)

The axisymmetric free transverse vibrations of sandwich annular plates of parabolically varying thickness are analysed retaining effects of transverse shear and rotatory inertia in the core of the plate. The equations of motion derived by Hamilton's energy principle are solved by cubic spline technique. Numerical results are computed for various values of parameters for first four normal modes of vibration.

1. INTRODUCTION

The sandwich construction is mostly used in aerospace industries and the use of variable thickness construction serves twofold requirements of safety and economy, but very few papers are available on the vibrations of circular sandwich plates. Those too are restricted to plates of uniform thickness. A survey of modern developments in the analysis of sandwich structures up to 1963 is given by Habip[1]. Yu and Koplik[2] has analysed the torsional vibrations of sandwich circular plates and spherical caps assuming the facings as membranes. Prasad and Gupta[3,4] studied the vibrations of circular sandwhich plates by retaining the effects of transverse shear and rotatory inertia for the core and the facings. The equations of motion were solved in terms of Bessel functions with the help of auxiliary variables. Mirza and Singh[5] have analysed the vibrations of circular sandwich plates assuming that the facings take bending but the core has no stresses parallel to facings. Equations of motion were solved in terms of Bessel functions.

It the present paper the axisymmetric transverse vibrations of sandwich annular plates of parabolically varying thickness are considered. The facings of the plate are taken of constant thickness and assumed as membranes but the core is taken of parabolically varying thickness and the effects of transverse shear and rotatory inertia are retained in it. The equations of motion are derived by Hamilton's energy principle and are solved by spline technique. The frequencies are obtained for the first four normal modes for various values of parameters for a plate clamped at the circular boundaries.

2. DISPLACEMENTS AND STRAINS

An annular sandwich plate of outer radius a, inner radius b and thickness $2h$ is considered. The thickness of the core and that of each of the facings is taken to be

$2h_1$ and h_2 respectively, so that $h = h_1 + h_2$. The thickness of the core varies with the radial distance whereas that of the facings remains constant. In cylindrical coordinates r, θ, z the axis of the plate is the line $r = 0$, the middle plane, the lower and the upper interfaces and the bottom and the top are planes $z=0$, $\mp h_1(r)$ and $\mp h(r)$, respectively. Both facings are of the same material which is taken different from that of the core. The various quantities for the core, the lowerf acing and the upper facing are distinguished from each other by the subscripts 1, 2 and 3, respectively.

As the facings of the plate are treated as membranes, the displacement components u_i, v_i and w_i, $i = 1, 2, 3$, in the directions of r, θ and z, respectively, are approximated by

$$\left.\begin{aligned} u_1(z, r, t) &= z\,\psi(r, t) \\ u_2(z, r, t) &= -u_3(z, r, t) = -h_1(r)\,\psi(r, t) \\ v_i(z, r, t) &= 0 \\ w_i(z, r, t) &= w(r, t) \end{aligned}\right\} \quad \ldots(1)$$

where ψ is the rotation in r, z-plane of the normal to the transverse section of the core and t is the time.

The nonzero strain components using expressions given by Love[6] are given by

$$\left.\begin{aligned} \epsilon_{r1} &= z\psi_{,r} \\ \epsilon_{r2} &= -\epsilon_{r3} = -h_{1,r}\,\psi - h_1\,\psi_{,r} \\ \epsilon_{\theta 1} &= (z/r)\,\psi \\ \epsilon_{\theta 2} &= -\epsilon_{\theta 3} = (-h_1/r)\,\psi \\ \epsilon_{rz1} &= \psi + w_{,r} \end{aligned}\right\} \quad \ldots(2)$$

where a comma followed by a suffixed variable denotes partial differentiation with respect to that variable. The strains ϵ_{rz2} and ϵ_{rz3} are put equal to zero as we assume the facings to be membranes.

3. Energy Considerations

The strain energies, W_i, of the core and the facings are given by

$$W_i = \tfrac{1}{2} \iiint [\sigma_{ri}\,\epsilon_{ri} + \sigma_{\theta i}\,\epsilon_{\theta i} + \sigma_{rzi}\,\epsilon_{rzi}]\, r\, d\theta\, dr\, dz \quad \ldots(3)$$

where σ_{ri}, etc. are the stress components. The limits of integrations are : from zero to 2π for θ and from b to a for r, both for the core and the facings. But the limits of integration for z are from $-h_1$ to h_1 for the core, from $-h$ to $-h_1$ for the lower facing and h_1 to h for the upper facing.

Substituting the values of strain components from equation (2) in equation (3), putting the respective limits of integrations and integrating, and then adding the three strain energies, the total energy, W_e, of the plate is obtained as

$$W_e = \pi \int_b^a [rP_1 \psi_{,r} + P_2\psi + rQ_r w_{,r}]\, dr \qquad \ldots(4)$$

where

$$\left.\begin{aligned}
&P_1 = M_{r1} - h_1 (N_{r2} - N_{r3}) \\
&P_2 = M_{\theta 1} - h_1 (N_{\theta 2} - N_{\theta 3}) - rh_{1,r} (N_{r2} - N_{r3}) + rQ_r \\
&(M_{r1}, M_{\theta 1}) = \int_{-h_1}^{h_1} (\sigma_{r1}, \sigma_{\theta 1})\, z\, dz \\
&(N_{r2}, N_{\theta 2}) = \int_{-h}^{-h_1} (\sigma_{r2}, \sigma_{\theta 2})\, dz \\
&(N_{r3}, N_{\theta 3}) = \int_{h_1}^{h} (\sigma_{r3}, \sigma_{\theta 3})\, dz \\
&Q_r = k_s \int_{-h_1}^{h_1} \sigma_{rz1}\, dz.
\end{aligned}\right\} \qquad \ldots(5)$$

Here k_s is the shear constant.

The first variation of the strain energy is

$$W_e = 2\pi \int_b^a [rP_1\, \delta(\psi_{,r}) + P_2\, \delta(\psi) + r\, Q_r\, \delta(W_{,r})]\, dr. \qquad \ldots(6)$$

The kinetic energies, T_l, of the core and the facings of the plate are given by

$$T_l = \frac{\rho_l}{2} \iiint [(u_{l,t})^2 + (w_{l,t})^2]\, r\, d\theta\, dr\, dz \qquad \ldots(7)$$

where ρ_l are the mass densities of the core and the facings and t is the time. The limits of integrations are the same as in the case of strain energies.

Substituting the values of u_l and w_l from eqn. (1) in eqns. (7), integrating and then adding the three kinetic energies, the total kinetic energy, T, of the plate is obtained as

$$T = 2\pi \int_b^a \left[\left(\frac{\rho_1 h_1^3}{3} + \rho_2 h_1^2 h_2\right) \psi_{,t}^2 + (\rho_1 h_1 + \rho_2 h_2)\, w_{,t}^2\right] r\, dt. \qquad \ldots(8)$$

The first variation of kinetic energy is given by

$$T = 2\pi \int_b^a \left[2\rho_1 \left\{\frac{h_1^3}{3} \psi_{,t}\, \delta(\psi_{,t}) + h_1 w_{,t}\, \delta(w_{,t})\right\} + 2\rho_2 \{h_1^2 h_2 \psi_{,t}\, \delta(\psi_{,t}) + h_2 w_{,t}\, \delta(w_{,t})\}\right] r\, dr. \qquad \ldots(9)$$

4. Equations of Motion

The equations of motion obtained by applying Hamilton's energy principle are

$$\left.\begin{aligned} &P_{1,r} + (1/r)\,P_1 - P_2 - \frac{2h_1^2}{3}(\rho_1 h_1 + 3\rho_2 h_2)\,\psi_{,tt} = 0 \\ &Q_{r,r} + (1/r)\,Q_r - 2\,(\rho_1 h_1 + \rho_2 h_2)\,w_{,tt} = 0. \end{aligned}\right\} \quad \ldots (10)$$

The stress-strain relations are taken to be

$$\left.\begin{aligned} &\sigma_{rl} = \lambda_l \epsilon_{rl} + \lambda_l' \epsilon_{\theta l}, \quad \sigma_{\theta l} = \lambda_l \epsilon_{\theta l} + \lambda_l' \epsilon_{rl} \\ &(\sigma_{r\theta l}, \sigma_{rzl}, \sigma_{\theta zl}) = \mu_l\,(\epsilon_{r\theta l}, \epsilon_{rzl}, \epsilon_{\theta zl}) \\ &\lambda_l = E_l/(1 - \nu_l^2), \quad \lambda_l' = \lambda_l \nu_l, \quad \mu_l = E_l/[2\,(1 + \nu_l)] \end{aligned}\right\} \quad \ldots (11)$$

where E_l and ν_l are Young's moduli and Poisson's ratios.

By using relations (2), (5) and (11), the equations of motion (10) reduce to the non-dimensional form

$$\left.\begin{aligned} &H_1^2 x^2 (H_1^2 R_1^2 + H_2 R_2)\,\psi_{,xx} + [3H_1 H_{1,x} x^2 (H_1 R_1 + 2H_2 R_2) \\ &\qquad + H_1^2 x\,(H_1 R_1 + H_2 R_2)]\,\psi_{,x} \\ &\qquad + [3H_1 H_{1,x} x\,(H_1 \nu_1 R_1 + H_2 R_2) + H_1^2\,(-H_1 R_1 - H_2 R_2) \\ &\qquad - 3k_s H_1 x]\,\psi - 3k_s H_1 x^2 \bar{W}_{,x} \\ &\qquad - x^2 (a^2 \rho_1/\lambda_1) H_1^2 R_1 (H_1 + 3H_2 R_\rho)\,\psi_{,tt} = 0, \\ &k_s H_{1,x}\,\psi_{,x} + k_s (H_{1,x} + H_1)\,\psi + k_s H_1 x \bar{W}_{,xx} + k_s (H_1 + H_{1,x} x) \\ &\bar{W}_{,x} - x^2 (a^2 \rho_1/\lambda_1) R_1 (H_1 + R_\rho H_2) \bar{W}_{,tt} = 0 \end{aligned}\right\} \quad \ldots (12)$$

where

$$\left.\begin{aligned} &x = r/a, \ (\bar{W} = w/a, \ H_1 = h_1/a, \ H_2 = h_2/a, \\ &R_\rho = \rho_2/\rho_1, \ R_1 = \lambda_1/\mu_1, \ R_2 = \lambda_2/\mu_1. \end{aligned}\right\} \quad \ldots (13)$$

For harmonic vibrations we take

$$\left.\begin{aligned} &\bar{W}(x, t) = W(x)\,e^{i\omega t} \\ &\psi(x, t) = G(x)\,e^{i\omega t} \end{aligned}\right\} \quad \ldots (14)$$

and for a core of parabolically varying thickness we take

$$H_1 = H_0 (1 - \beta x^2) \quad \ldots (15)$$

where β is the taper constant.

Substituting eqns. (14) and (15) in eqns (12) we get the equations of motion as

$$\left.\begin{aligned}
&(A_0 x^6 + A_1 x^4 + A_2 x^2)\frac{d^2G}{dx^2} + (A_3 x^5 + A_4 x^3 + A_2 x)\frac{dG}{dx} \\
&\quad + (A_5 x^4 + A_6 x^2 - A_2) G + A_7 x^2 \frac{dW}{dx} \\
&\quad + \Omega^2 (A_0 x^6 + A_8 x^4 + A_9 x^2) G = 0 \\
&(A_{10} x^3 + A_{11} x)\frac{dG}{dx} + (A_{12} x^2 + A_{11}) G + (A_{10} x^3 + A_{11} x)\frac{d^2W}{dx^2} \\
&\quad + (A_{12} x^2 + A_{11})\frac{dW}{dx} + \Omega^2 (A_{13} x^3 + A_{14} x) W = 0
\end{aligned}\right\} \quad \ldots(16)$$

where

$$\left.\begin{aligned}
&A_0 = H_0^2 \beta^2 R_1, A_1 = H_0 \beta (2H_0 R_1 + 3H_2 R_2) \\
&A_2 = H_0 (H_0 R_1 + 3H_2 R_2), A_3 = 7H_0^2 \beta^2 R_1 \\
&A_4 = -H_0 \beta (8H_0 R_1 + 15H_2 R_2), A_5 = H_0^2 \beta^2 R_1 (6\nu_1 - 1) \\
&A_6 = 2H_0^2 \beta R_1 (1 - 3\nu_1) - 9H_0 H_2 \beta R_2 - 3K_s \\
&A_7 = -3k_s, A_8 = -H_0 \beta R_1 (2H_0 + 3H_2 H_\rho) \\
&A_9 = H_0 R_1 (H_0 + 3H_2 R_\rho), A_{10} = -k_s H_0 \beta \\
&A_{11} = k_s H_0, A_{12} = -3k_s H_0 \beta, A_{13} = -H_0 \beta R_1 \\
&A_{14} = R_1 (H_0 + R_\rho H_2), \Omega^2 = \rho_1 a^2 \omega^2 / \lambda_1.
\end{aligned}\right\} \quad \ldots(17)$$

Here Ω is a nondimensional frequency parameter.

5. Solution by A Cubic Spline Technique

Let $f(x)$ and $g(x)$ be two functions with continuous derivatives in the range $(b/a, 1)$. Choose $(n+1)$ points $x_0, x_1, x_2, \ldots, x_n$ in the range $b/a \leqslant x \leqslant 1$ such that $b/a = x_0 < x_1 < x_2 < \ldots < x_n = 1$.

Let the approximating functions $W(x)$ for $f(x)$ and $G(x)$ for $g(x)$ be cubic splines satisfying the following the properties;

(a) $W(x)$ and $G(x)$ are cubic polynomials in each interval (x_k, x_{k+1});

(b) $W(x_k) = f(x_k)$ and $G(x_k) = g(x_k)$, $k = O(1) n$;

(c) $W(x)$ and $G(x)$ have continuous derivatives up to second order.

According to above properties cubic splines take the form

$$\left.\begin{aligned}
W(x) &= a_0 + \sum_{i=1}^{2} a_i (x - x_0)^i + \sum_{j=0}^{n-1} b_j (x - x_j)^3_+, \\
G(x) &= c_0 + \sum_{i=1}^{2} c_i (x - x_0)^i + \sum_{j=0}^{n-1} d_j (x - x_j)^3_+.
\end{aligned}\right\} \quad \ldots(18)$$

Here

$$(x - x_j)_+ = \begin{cases} 0 & \text{if } x \leqslant x_j \\ x - x_j & \text{if } x > x_j \end{cases} \qquad \ldots(19)$$

$a_0, a_1, a_2, b_0, \ldots, b_{n-1}, c_0, c_1, c_2, d_0, \ldots, d_{n-1}$ are $2(n+3)$ unknown constants. For simplicity we assume that knots x_i to be equally spaced in the interval $(x_0, 1)$ with distances $\Delta x = (1 - x_0)/n$, then

$$x_j = x_0 + j\,\Delta x\ ,\ j = 1, 2, \ldots, n. \qquad \ldots(20)$$

The satisfaction of differential equations (16) by collocation of $(n+1)$ knots in the interval $(x_0, 1)$ together with the boundary conditions give the required number of equations to determine the unknown constants.

On substitution of $W(x)$, $G(x)$ and their derivatives from equations (18) into equations (16) to satisfy the equations at the m^{th} knot, we get

$$\left.\begin{aligned} & a_0[0] + a_1[B_5] + a_2[2B_3(x_m - x_0)] + c_0[B_2] + c_1[B_2(x_m - x_0) + B_1] \\ & \quad + c_2[B_2(x_m - x_0)^2 + 2B_1(x_m - x_0) + 2B_0] \\ & \quad + \sum_{j=m+1}^{n-1} b_j[3B_3(x_m - x_j)^2] + \sum_{j=m+1}^{n-1} d_j[B_2(x_m - x_j)^3 \\ & \quad + 3B_1(x_m - x_j)^2 + 6B_0(x_m - x_j)] = 0 \\ & a_0[B_6] + a_1[B_6(x_m - x_0) + B_5] + a_2[B_6(x_m - x_0)^2 + 2B_5(x_m - x_0) + 2B_4] \\ & \quad + c_0[B_5] + c_1[B_5(x_m - x_0) + B_4] + c_2[B_5(x_m - x_0)^2 \\ & \quad + 2B_4(x_m - x_0)] + \sum_{j=m+1}^{n-1} b_j[B_6(x_m - x_j)^3 + 3B_5(x_m - x_j)^2 \\ & \quad + 6B_4(x_m - x_j)] + \sum_{j=m+1}^{n-1} d_j[B_5(x_m - x_j)^3 \\ & \quad + 3B_4(x_m - x_j)^2] = 0 \end{aligned}\right\} \qquad \ldots(21)$$

where

$$\left.\begin{aligned} B_0 &= A_0 x_m^6 + A_1 x_m^4 + A_2 x_m^2 \\ B_1 &= A_3 x_m^5 + A_4 x_m^3 + A_2 x_m \\ B_2 &= A_5 x_m^4 + A_6 x_m^2 - A_2 + \Omega^2 x_m^2 (A_2 x_m^4 + A_8 x_m^2 + A_9) \\ B_3 &= A_7 x_m^2\ ,\ B_4 = A_{10} x_m^3 + A_{11} x_m \\ B_5 &= A_{12} x_m^2 + A_{11},\ B_6 = \Omega_2 (A_{13} x_m^3 + A_{14} x_m). \end{aligned}\right\} \qquad \ldots(22)$$

Thus we have a set of 2 $(n + 1)$ equations in terms of 2 $(n + 3)$ unknown constants $a_0, a_1, a_2\ b_0,\ldots,b_{n-1}, c_0\ c_1, c_2, d_0,\ldots, d_{n-1}$ which can be written in matrix form as

$$[A]\,[C] = [0] \qquad \ldots(23)$$

where A and C are matrices of order $2\,(n + 1) \times 2\,(n + 3)$ and $2\,(n + 3) \times 1$ respectively.

Let the plate be clamped at both circular boundaries. Then the boundary conditions are

$$W(x) = G(x) = 0 \text{ at } x = x_0 \text{ and } 1. \qquad \ldots(24)$$

Applying the boundary conditions (24) we get a set of four homogeneous equations in terms of $a_0, a_1, a_2, b_0,\ldots,b_{n-1}, c_0, c_1, c_2, d_0,\ldots,b_{n-1}$, which can be expressed in matrix form as

$$[B]\,[C] = [0] \qquad \ldots(25)$$

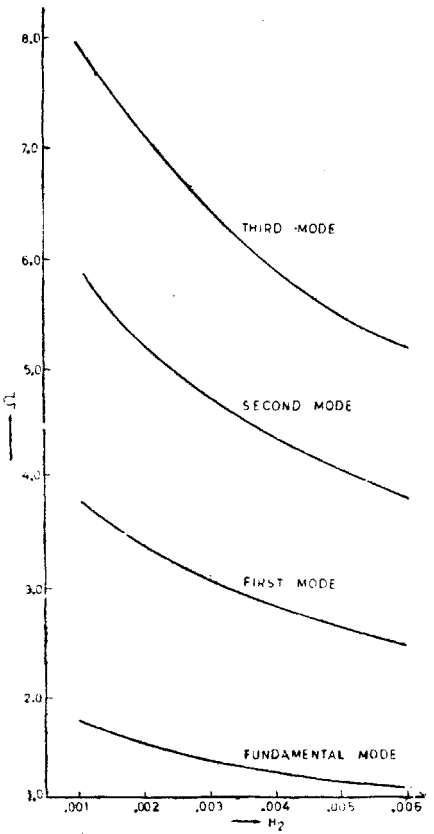

FIG. 1. Ω *VS.* H_2 for $H_0 = .10$, $\beta = 0.3$ and $X_0 = .10$

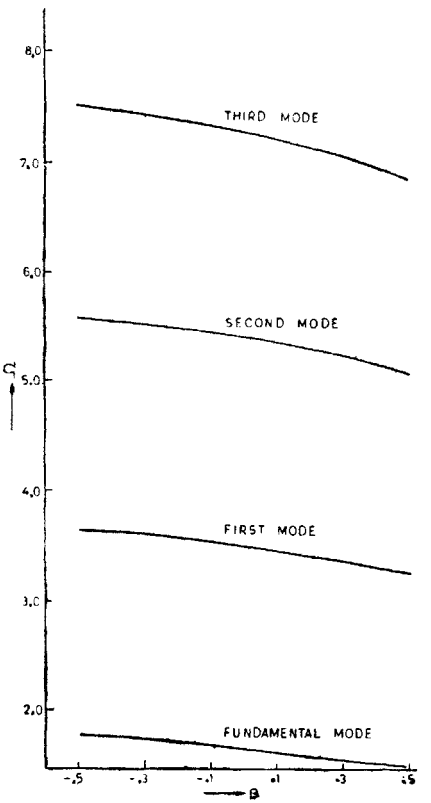

FIG. 2. Ω *VS* β for $H_0 = .10$, $H_2 = .002$ and $X_0 = .10$

where B is a matrix of order $4 \times 2(n + 3)$.

Equations (25) together with the field equations (23) constitute a complete set of $2(n + 3)$ equations, which can be written as

$$\left[\frac{A}{B}\right][C] = [0]. \qquad \ldots(26)$$

For a nontrivial solution of equations (26) we must have

$$\left|\frac{A}{B}\right| = 0. \qquad \ldots(27)$$

6. Numerical Results and Discussion

For purposes of numerical studies, k_s can be taken equal to unity as was done by Koplik and Yu[2]. The materials for the core and the facings are taken to be cellular cellulose acetate and aluminium respectively. The variou sconstants for these materials are $R_1 = 2.20$, $R_2 = 4790.0$, $R_\rho = 34.4$ and $\nu_1 = .0909091$.

The frequency parameter Ω has been computed for an annular plate clamped at both circular boundaries by taking $n = 20$. Higher values of n do not improve the results except in the third or fourth significant digits.

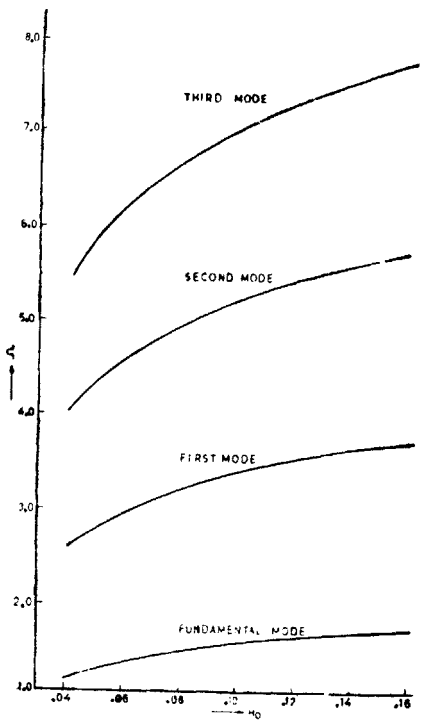

Fig. 3. Ω *VS* H_0 for $\beta = 0.3$ $H_2 = .002$ and $X_0 = .10$

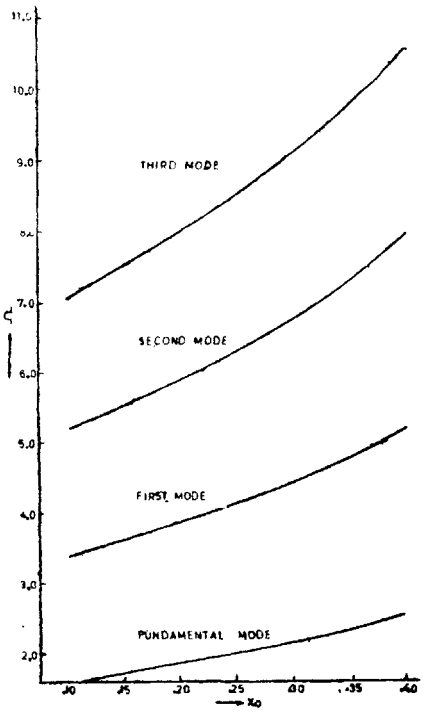

Fig. 4. Ω *VS.* X_0 for $H_0 = .10$, $H_2 = .002$ and $X_0 = .10$

The frequency parameter Ω for the first four normal modes of vibration for various values of H_2, β, H_0 and x_0 are plotted in Figs. 1 to 4. Figure 1 shows that Ω decreases with an increase in thickness of the facings in all the modes of vibration, first rapidly and then slowly and decreases more in higher modes of vibrations than in lower modes of vibrations. Figure 2 shows that Ω decreases with increase in the value β in all the modes of vibrations. But it decreases first rapidly and then slowly and the change is faster in higher modes of vibrations than in the case of lower modes of vibrations. This is due to the fact that as β increases the average thickness of plate decreases. From Fig. 3 we conclude that Ω increases in all the modes of vibrations for increasing values of H_0 first rapidly and then slowly. This can be attributed to the fact that as H_0 increases the stiffness of the plate increases. Figure 4 shows that Ω first increases slowly and then rapidly with an increase in inner radius of the plate for all the modes of vibrations. This again is due to an increase in the stiffness of the plate with increasing x_0.

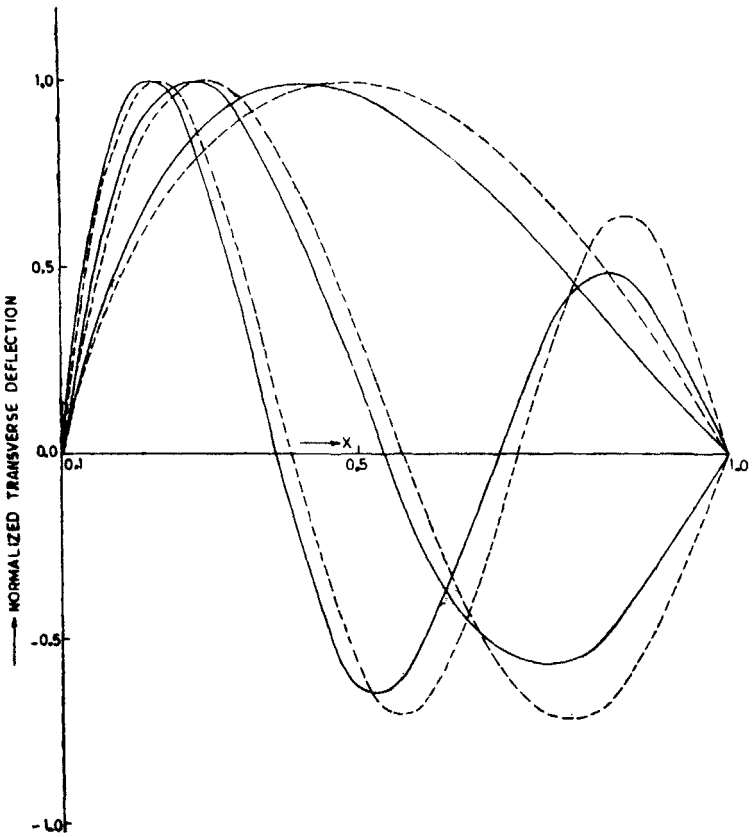

FIG. 5. Normalized Transverse Deflections in the First Three Modes of vibration for $H_0 = .10$; $H_2 = .002, X_0 = .10; \beta = -.5, \ldots, \beta = .5$

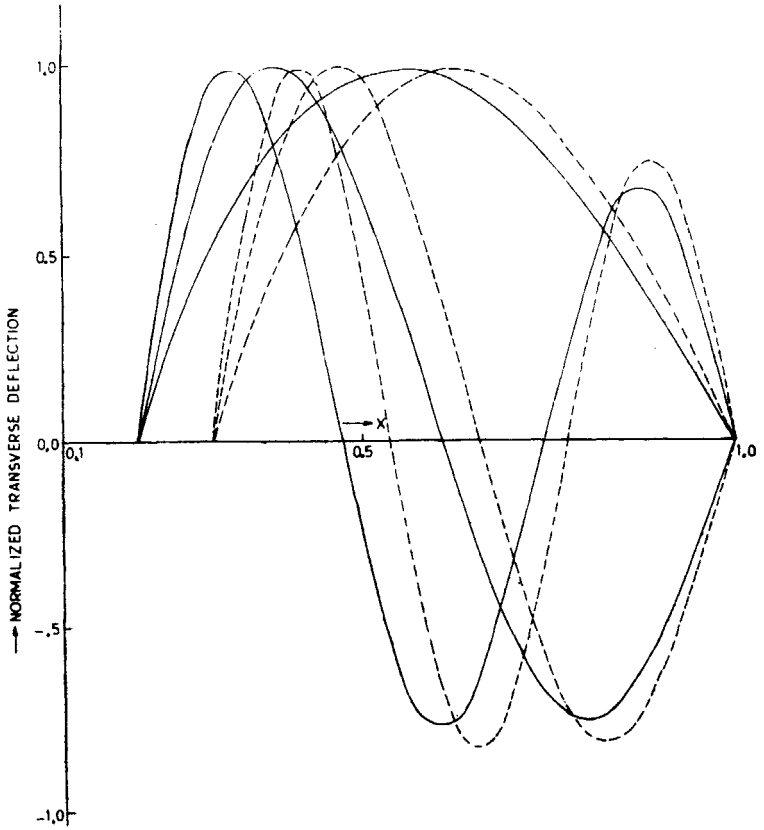

FIG. 6 Normalized transverse deflections in the first three Modes of vibration for $H_0 = .10$, $H_2 = .002$, $\beta = 0.3$; —— $X_0 = .2, \cdots, X_0 = .3$

The normalized transverse deflections for the first three modes of vibration are plotted in Fig. 5 for $\beta = -0.5, 0.5$ and in Fig. 6 for $x_0 = 0.2$ and 0.3. Figure 5 shows the shift of nodal circles towards the outer edge of the plate and also that the deflections increase near the outer edge as β varies from negative to positive values. This is due to the fact that the plate becomes thinner towards the outer edge with the increase in value of β. Figure 6 shows the shift of nodal circles and increase in deflections towards the outer edge when x_0 varies from 0.2 to 0.3. This is obvious because the inner radius of the plate is increased.

REFERENCES

1. L. M. Habip, *Appl. Mech. Reviews*, 18 (1965), 93–98.
2. Y. Y. Yu, and B. Koplik, *J. Appl. Mech.* 34 (1967), 787–89.
3. C. Prasad, and A. P. Gupta, *Proc. Nat. Acad. Sci. India* 48 (1968).
4. C. Prasad, and A. P. Gupta, *J. Pure & Appl. Math.* 14 (1979), 1002–08.
5. S. Mirza, and A. V. Singh, *J. Engng. Math.* 8 (1974), 71–79.
6. A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*. Dover Publication, New York, 1944.