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SINGULAR INTEGRAL EQUATION OF THE FIRST KIND WITH A CAUCHY KERNEL—REVISITED

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This note is concerned with deriving the solution of a singular integral equation of the first kind with a Cauchy kernel in a finite interval by a formal but direct method which appears to be simple and straightforward compared to other methods known in the literature.

1. INTRODUCTION

One of the standard methods for solving the integral equation (IE) of the first kind with a Caucy kernel

$$\int_0^1 \frac{\varphi(\tau)}{\tau - t} d\tau = \psi(t), \; 0 < t < 1 \qquad \ldots(1.1)$$

where the integral is in the sense of Cauchy principal value, involves the so called function theoretic method wherein the I E is reduced to a Riemann Hilbert boundary value problem for a sectionally analytic function[3]. By this method[3] (p. 427) solution of (1.1) has three different forms depending upon the requirement regarding the behaviour of the function $\varphi(t)$ at the end points. When $\varphi(t)$ is unbounded at both the ends, the solution is

$$\varphi(t) = -\frac{1}{\{t(1-t)\}^{1/2}}\left[\frac{1}{\pi}\int_0^1 \{\tau(1-\tau)\}^{1/2}\Psi(\tau)\frac{d\tau}{\tau - t} + c\right] \qquad \ldots(1.2)$$

where c is an arbitrary constant. When $\varphi(t)$ is bounded at $t = 0$ and unbounded at $t = 1$, the solution is

$$\varphi(t) = -\left(\frac{t}{1-t}\right)^{1/2}\frac{1}{\pi}\int_0^1\left(\frac{1-\tau}{\tau}\right)^{1/2}\Psi(\tau)\frac{d\tau}{\tau - t}\left(\frac{1-\tau}{\tau}\right)^{1/2} \qquad \ldots(1.3)$$

When $\tau(t)$ is bounded at both the ends, the solution exists only when the following solvability condition

$$\int_0^1 \frac{\Psi(\tau)}{\{\tau(1-\tau)\}^{1/2}} d\tau = 0 \qquad \ldots(1.4)$$

is satisfied, and in that case, the solution is

$$\varphi(t) = -\{t(1-t)\}^{1/2} \frac{1}{\pi} \int_0^1 \frac{\Psi(\tau)}{\{\tau(1-\tau)\}^{1/2}} \frac{d\tau}{\tau - t}. \qquad \ldots(1.5)$$

In the present note we obtain the solution of (1.1) by first reducing the Cauchy kernel to a logarithmic kernel and then exploiting the simple integral

$$\ln\left|\frac{x-y}{x+y}\right| = -2 \int_0^{\min(x,y)} \frac{u\, du}{\{(y^2-u^2)(x^2-u^2)\}^{1/2}} \qquad \ldots(1.6)$$

the IE is reduced to a pair of Volterra I E's, of first kind whose solutions are obtained following Copson[1]. The solution is first obtained in the form (1. 3) from which the forms (1.5) and (1.2) are deduced under appropriate behaviours of φ at the end points.

Although this method appears to be not very much different from that used in Peters[4] and also in Estrada and Kanwal[2], it appears to be simple and straightforward to merit further illustration.

2. Reduction to a Pair of Volterra Integral Equations of the First Kind

Substituting $\tau = x^2$, $t = y^2$ where $x > 0$, $y > 0$, (1. 1) can be written as

$$\frac{1}{\pi} \int_0^1 f(x) \left(\frac{1}{x-y} + \frac{1}{x+y}\right) dx = \Psi_1(y), \quad 0 < y < 1 \qquad \ldots(2.1)$$

where $f(x) \equiv \varphi(x^2)$, $\Psi_1(y) \equiv \Psi(y^2)$. $\ldots(2.2)$

Integration with respect to y reduces (2.1) to

$$-\frac{1}{\pi} \int_0^1 f(x) \ln\left|\frac{x-y}{x+y}\right| dx = g(y), \quad 0 < y < 1 \qquad \ldots(2.3)$$

where $g(y) = \int_0^y \Psi_1(y)\, dy$. $\ldots(2.4)$

Using the result (1.6) in (2.3), after simplification it reduces to the pair of Volterra IE's of first kind given by

$$\int_u^1 \frac{f(x)}{(x^2-u^2)^{1/2}}\,dx = F(u),\ 0<u<1 \qquad \ldots(2.5)$$

and

$$\int_0^y \frac{u\,F(u)}{(y^2-u^2)^{1/2}}\,du = \frac{\pi}{2}\,g(y),\ 0<y<1. \qquad \ldots(2.6)$$

3. Solution of the Integral Equation

Following Copson[1] and using (2.4) the solution of (2.6) is

$$F(u) = \int_0^u \frac{\Psi_1(y)}{(u^2-y^2)^{1/2}}\,dy. \qquad \ldots(3.1)$$

Again, following Copson[1], solution of (2.5) is

$$f(x) = \frac{2}{\pi}\,\frac{x}{(1-x^2)^{1/2}}\,F(1) - \frac{2}{\pi}\,x\int_x^1 \frac{F'(u)}{(u^2-x^2)^{1/2}}\,du. \qquad \ldots(3.2)$$

Using (3.1) the integral in the right side of (3.2) becomes after simplification

$$\int_0^1 y\,\Psi_1'(y)\left[\int_{\max(x,y)}^1 \frac{u\,du}{\{(u^2-x^2)(u^2-y^2)\}^{1/2}}\right]dy$$

$$= \frac{1}{x}\int_0^1 \Psi_1'(y)\ln\frac{x(1-y^2)^{1/2}+y(1-x^2)^{1/2}}{(|x^2-y^2|)^{1/2}}\,dy$$

$$= -(1-x^2)^{1/2}\int_0^1 \frac{\Psi_1(y)}{(1-y^2)^{1/2}}\,\frac{dy}{x^2-y^2}. \qquad \ldots(3.3)$$

Thus using (3.1) and (3.3) in (3.2) and substituting back $x^2=\tau$, $y^2=t$, we arrive at

$$\varphi(t) = \left(\frac{t}{1-t}\right)^{1/2}\frac{1}{\pi}\int_0^1 \frac{\Psi(\tau)}{\{\tau(1-\tau)\}^{1/2}}\,d\tau$$

(equation continued on p. 1343)

$$-\{t(1-t)\}^{1/2}\frac{1}{\pi}\int_0^1 \frac{\Psi(\tau)}{\{\tau(1-\tau)\}^{1/2}}\frac{d\tau}{\tau-t} \qquad \ldots(3.4)$$

$$= -\left(\frac{t}{1-t}\right)^{1/2}\frac{1}{\pi}\int_0^1 \left(\frac{1-\tau}{\tau}\right)^{1/2}\Psi(\tau)\,\frac{d\tau}{\tau-t} \qquad \ldots(3.5)$$

which is (1.8) given earlier.

The form (3.5) is useful when $\varphi(t)$ is bounded at $t = 0$ but unbounded at $t = 1$. If however, $\varphi(t)$ is required to be bounded at both the end points, we note from (3.4) that such a solution exists only when the solvability condition (1.4) is satisfied, and in that case the solution reduces to (1.5).

Again, to deduce the from of solution which is suitable when $\varphi(t)$ is required to be unbounded at both the ends, we note that, using (3.5) we obtain

$$\int_0^1 \varphi(t)\,dt = \int_0^1 \left(\frac{1-\tau}{\tau}\right)^{1/2}\Psi(\tau)\left[\frac{1}{\pi}\int_0^1 \left(\frac{t}{1-\tau}\right)^{1/2}\frac{dt}{t-\tau}\right]d\tau$$

$$= \int_0^1 \left(\frac{1-\tau}{\tau}\right)^{1/2}\Psi(\tau)\,d\tau \qquad \ldots(3.6)$$

as the inner integral is unity. Then using (3.5) and (3.6) we can write

$$\varphi(t) - \frac{1}{\{t(1-t)\}^{1/2}}\,\frac{1}{\pi}\int_0^1 \varphi(\tau)\,d\tau$$

$$= -\left[\left(\frac{t}{1-t}\right)^{1/2}\frac{1}{\pi}\int_0^1 \left(\frac{1-\tau}{\tau}\right)^{1/2}\Psi(\tau)\,\frac{d\tau}{\tau-t}\right.$$

$$\left. -\{t(1-t)\}^{1/2}\int_0^1 \left(\frac{1-\tau}{\tau}\right)^{1/2}\Psi(\tau)\,d\tau\right]$$

$$= -\frac{1}{\{t(1-t)\}^{1/2}}\,\frac{1}{\pi}\int_0^1 \{\tau(1-\tau)\}^{1/2}\Psi(\tau)\,\frac{d\tau}{\tau-t}\,.$$

Writing $c = -\frac{1}{\pi}\int_0^1 \varphi(\tau)\,d\tau$ we finally obtain

$$\varphi(t) = -\frac{1}{\{t(1-t)\}^{1/2}}\left[\frac{1}{\pi}\int_0^1 \{\tau(1-\tau)\}^{1/2}\,\Psi(\tau)\,\frac{d\tau}{\tau - t} + c\right]. \qquad \ldots(37)$$

But in view of the result

$$\int_0^1 \frac{1}{\{\tau(1-\tau)\}^{1/1}}\,\frac{d\tau}{\tau - t} = 0,\quad 0 < t < 1 \qquad \ldots(3.8)$$

c in (3.7) has the character of an arbitrary constant. Thus (1.2) is deduced.

4. Conclusion

The singular integral equation (1.1) is solved by a direct method which seems to be simple and straightforward. The three different forms of solution appropriate for three different types of behaviours of the function at the end points are obtained by this method. Also the function $\Psi(t)$ here is assumed to satisfy the appropriate necessary conditions which make the different integrals involving $\Psi(t)$ to be convergent.

References

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