Indian J. pure appl. Math., 17 (12) : 1432-1437, December 1986

FIRST-ORDER PERTURBATIONS OF THE THREE FINITE BODY PROBLEM

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(*Received* 13 *January* 1986; *after revision* 7 *October* 1986)

This investigation deals with the general motion of a triaxial rigid body under the mutual gravitational attraction force of two spherical rigid bodies. Using Delaunay-Andoyer's elements, we obtain the disturbing function in a suitable form. We use the torque-free motion and the Keplerian motions as unperturbed motions. The first order perturbations are derived by using the variation method of the arbitrary constants of the unperturbed motions.

1. STATEMENT OF THE PROBLEM

Let us consider the translational-rotational of a triaxial rigid body T_1 in the gravitational field of two spheres T_0 and T_2 with spherical density distributions. Let $Oxyz$ be a system of coordinates with origin at the center of mass of T_0 and with axes of constant orientation. Let $O'\,x'\,y'\,z'$ be a system of coordinates with origin at the center of mass of bodies T_0 and T_1 and with axes which are parallel to the corresponding axes of system $Oxyz$, and let $O_1\,\xi\,\eta\,\zeta$ be the principal axes of the body T_1 having the origin at its center of mass. The symbol m_i denotes the mass of the body T_i ($i = 0$, 1, 2) while the symbols A, B and C will be used to denote the principal centeral moments of inertia of the body T_1.

2. EQUATIONS OF MOTION

Let us describe the translational motion of the center of mass of T_1 relative to the center of T_0 by Delaunay elements L_1, G_1, H_1, l_1, g_1 and h_1, while the translational motion of the body T_2 relative to the center of mass of the bodies T_0 and T_1 be described by the Delaunay elements L_2, G_2, H_2, l_2, g_2 and h_2.

The Delaunay's elements are defined as :

$$\begin{aligned} L_j &= (\mu_j\, a_j)^{1/2} &&, \; l_j = \text{mean anomaly} \\ G_j &= [\mu_j\, a_j\,(1 - e_j^2)]^{1/2} &&, \; g_j = \text{argument of pericenter} \\ H_j &= G_j \cos i_j &&, \; h_j = \text{longitude of ascending node} \end{aligned} \qquad \ldots(1)$$

$$\mu_1 = f\, m_0^2 \;\; m_1^2 /(m_0 + m_1)\,,\; \mu_2 = f m_0\,(m_0 + m_1)\;\; m_2^2 \;/(m_0 + m_1 + m_2)$$

and f is the gravitational constant, a_j, e_j are the semi-major axis and eccentricity of the orbit of the corresponding body T_j $(j = 1, 2)$, i_j-the angle between the orbital plane T_j and the reference plane xy.

The rotational motion of the body T_1 can be described by Andoyer's elements L, G, H, l, g, h which are defined as[2,3]

G : the angular momentum vector due to the rotational of the body T_1,

L : ζ-component of the angular momentum,

H : Z-component of the angular momentum, l, g, h are the conjugate angles of the elements L, G, H.

The equations of the general motion in the Delaunay-Andoyer's elements can be written in the canonical form

$$\frac{d}{dt}(L, G, H, L_i, G_i, H_i) = \frac{-\partial F}{\partial (l, g, h, l_i, g_i, h_i)}$$

$$\frac{d}{dt}(l, g, h, l_i, g_i)\, h_i, = \partial \frac{\partial F}{(L, G, H, L_i, G_i, H_i)} \qquad \ldots(2)$$

$$(i = 1, 2),$$

where

$$F = F_0 + F_1 \qquad \ldots(3)$$

$$F_0 = -\frac{\mu_1^2}{2\nu_1 L_1^2} - \frac{\mu_2^2}{2\nu_2 L_2^2} + \frac{(G^2 - L^2)}{2}\left(\frac{\sin^2 l}{A} + \frac{\cos^2 l}{B}\right)$$

$$+ \frac{L^2}{2C},$$

$$\nu_1 = m_0 m_1/(m_0 + m_1), \quad \nu_2 = \frac{m_2 (m_0 + m_1)}{(m_0 + m_1 + m_2)},$$

$$F_1 = \sum_{k_1 \cdots k_5} \left(\frac{\lambda_0}{a_1^3}\right) u_{k_1 \ldots k_5} \cos [k_1 l_1 + k_2 g_1 + k_3 (h - h_1) + k_4 g + k_5 l]$$

$$+ \frac{3}{32} \frac{f m_0 m_1 m_2}{(m_0 + m_1)} \sum_{k_1 \cdots k_5} \left(\frac{a_1^2}{a_2^3}\right) V_{k_1 \cdots k_5} \cos [k_1 l_1 + k_2 l_2$$

$$+ k_3 g_1 + k_4 g_2 + k_5 (h_1 - h_2)]$$

where

$$u_{k_1,\,2\nu,\,2\epsilon,\,2\delta,\,2\sigma} = -\frac{3}{256}\,(1+\delta\,\sigma\cos\theta)^2\,(1+\epsilon\,\delta\cos\rho)^2\,(1-\nu\,\epsilon\cos i_1)^2 \times\left(C_{k_1}^{-3,2}+\nu\,S_{k_1}^{-3,2}\right),$$

$$u_{k_1,\,2\nu,\,2\epsilon,\,\delta,\,2\sigma} = \frac{3}{256}\,(\sin 2\theta+2\delta\sigma\sin\theta)\,(\sin 2\rho+2\epsilon\delta\sin\rho) \times(1-\nu\,\epsilon\cos i_1)^2\left(C_{k_1}^{-3,2}+\nu\,S_{k_1}^{-3,2}\right)$$

$$u_{k_1,\,0,\,0,\,0,\,0} = \left[\frac{3}{16}(V+2\,\mu W)-(1+\mu)\right]C_{k_1}^{-3,0}\,,$$

$$\lambda_0 = \frac{-f}{2}\,m_0\,(B-A)\,,\quad \mu=(C-A)/(B-A)$$

$$V = (1+\cos^2 i_1)\,[(1+\cos^2\theta)\,(1+\cos^2\rho)+2\sin^2\theta\sin^2\rho] + 2\sin^2 i_1\,[\sin^2\rho\,(1+\cos^2\theta)+2\sin^2\theta\cos^2\rho]$$

$$W = (1+\cos^2\theta)\,[\sin^2\theta\,(1+\cos^2\rho)+2\cos^2\theta\sin^2\rho] + 2\sin^2 i_1\,[\sin^2\theta\,\sin^2\rho+2\cos^2\theta\cos^2\rho]\,,$$

$k_1\,(0, 1, \ldots)$; $\nu, \epsilon\,\delta, \sigma = \pm 1$; $\cos\theta = L/G$; $\cos\rho = H/G$;

$C_k^{m,n}$, $S_k^{m,n}$ are functions of orbital eccentricity e_j which defined in[4], and the other coefficients $V_{k_1\ldots k_5}$ can be defined

$$V_{k_1\,k_2,\,2\epsilon,\,2\delta,\,2\sigma} = \frac{1}{8}(1+\epsilon\,\delta\cos i_1)^2\,(1-\delta\sigma\cos i_2)^2 \times\left(C_{k_1}^{2,2}+\epsilon\,S_{k}^{2,2}\right).\left(C_{k_2}^{-3,2}+\nu\,\delta\,S_{k_2}^{-3,2}\right)$$

$$V_{k_1,\,k_2,\,2\epsilon,\,2\delta,\,\sigma} = \frac{1}{8}\,(sm\,2\,i_1+2\,\epsilon\,\sigma\sin i_1)\,(\sin 2i_2-2\,\delta\sigma\sin i_2) \times\left(C_{k_1}^{2,2}+\epsilon\,S_{k_1}^{2,2}\right)\left(C_{k_2}^{-3,2}+\nu\delta\,S_{k_2}^{-3,2}\right)$$

$$V_{k_1,\,k_2,\,0,\,0,\,0} = \left[2\sin^2 i_1\sin^2 i_2+(1+\cos^2 i_1)\,(1+\cos^2 i_2) - \frac{16}{3}\right] \times C_{k_1}^{2,0}\,C_{k_2}^{-3,0}\,,$$

where

$$k_1, k_2\ (0,\ 1, \ldots).$$

3. The Unperturbed Motions

In the unperturbed motions, we consider the Hamiltonian function F_0. After substituting in eqns. (2) and integrating we get the elements in the form[5],

$$
\begin{aligned}
L_i &= L_{i0} = \text{const.} \quad , & l_i &= s_i t + l_{i0} \\
G_i &= G_{i0} = \text{const.} \quad , & g_i &= g_{i0} = \text{const.} \\
H_i &= H_{i0} = \text{const.} \quad , & h_i &= h_{i0} = \text{const.} \\
L &= L_0 = \text{const.} \quad , & l &= q_1 t + l_0 \\
G &= G_0 = \text{const.} \quad , & g &= q_2 t + g_0 \\
H &= H_0 = \text{const.} \quad , & h &= h_0 = \text{const.}
\end{aligned}
\qquad \ldots(4)
$$

where

$$q_1 = \frac{\partial \bar{F}_0}{\partial G},\ q_2 = \frac{\partial \bar{F}_0}{\partial I},\ s_i = \frac{\mu_i^2}{\nu_i L_i^2}\ (i = 1, 2),$$

$$\bar{F}_0 = \frac{G^2}{2A}\left(1 - \frac{C - A}{C}\frac{x^2}{x^2 + \lambda_n^2}\right)$$

$$I = \frac{1}{2\pi} \oint L\, dl$$

$$= \frac{2G}{x}\left(\frac{1 + x^2}{\lambda^2 + x^2}\right)^{1/2} [\,(\lambda^2 + x^2)\, \Pi\, (\tfrac{1}{2}\pi,\ x^2, \lambda) - \lambda^2 K(\lambda)],$$

$$x^2 = \frac{C(B - A)}{A(C - B)},\ \lambda^2 = x^2\left(\frac{2\,CH - G^2}{G^2 - 2AH}\right).$$

K-ellipitic function of the first type and Π-elliptic function of the third type. The results of unperturbed motions show that the eccentricity, the semi-major axis of the orbit and also the angular momentum vector of the rotation of T_1 remain unaltered.

4. The Perturbed Motions

Using the variation method[1] of the arbitrary constant (4) of the unperturbed motions, the equations of the perturbed motions can be written in the form :

$$\frac{d}{dt}(l_{i0}, g_{i0}, h_{i0},\ l_0, g_0, h_0) = \frac{\partial F_{10}}{\partial (L_{i0}, G_{i0}, H_{i0}, L_0, G_0, H_0)},$$

$$\frac{d}{dt}(l_{i0}, G_{i0},\ H_{i0}, L_0, G_0, H_0) = \frac{-\partial F_{10}}{\partial (l_{i0}, g_{i0}, h_{i0}, l_0\ g_0\ h_0)},$$

where F_{10} is the function F_1 after substituting by the elements of the uperturbed motions. Equations (5) can be integrated when we consider the mass of the triaxial body T_1 is very small in comparison with the two spherical rigid bodies and also the value $\lambda_0 = A - B$ is small. These suppositions agree with the problem of the triaxial Earth satellite under the perturbation of the Moon (Sun). The elements of the first order perturbations of eqns. (5) are in the form :

$$l_{t0} = \frac{\partial F'_{10}}{\partial L_{t0}} t + \int \frac{\partial F_1^*}{\partial L_{t0}} dt, \; l_0 = \frac{\partial F'_{10}}{\partial L_0} t + \int \frac{\partial F_1^*}{\partial L_0} dt,$$

$$g_{t0} = \frac{\partial F'_{10}}{\partial G_{t0}} t + \int \frac{\partial F_1^*}{\partial G_{t0}} dt, \; g_0 = \frac{\partial F'_{10}}{\partial G_0} t + \int \frac{\partial F_1^*}{\partial G_0} dt,$$

$$h_{t0} = \frac{\partial F'_{t0}}{\partial H_{t0}} t + \int \frac{\partial F_1^*}{\partial H_{t0}} dt, \; h_0 = \frac{\partial F'_{10}}{\partial H_0} t + \int \frac{\partial F_1^*}{\partial H_0} dt$$

$$L_{t0} = -\int \frac{\partial F_1^*}{\partial l_{t0}} dt \quad , \quad L_0 = -\int \frac{\partial F_1^*}{\partial l_0} dt,$$

$$G_{t0} = -\int \frac{\partial F_1^*}{\partial g_{t0}} dt \quad , \quad G_0 = -\int \frac{\partial F_1^*}{\partial g_0} dt,$$

$$H_{t0} = -\int \frac{\partial F_1^*}{\partial h_{t0}} dt \quad , \quad H_0 = -\int \frac{\partial F_1^*}{\partial h_0} dt,$$

where

$$F_{10} = \frac{\lambda}{a_1^3} u_{0\cdot0\cdot0\cdot0\cdot0} + \frac{3}{32} \frac{f\, m_0 m_1 m_2}{(m_0 + m_1)} \frac{a_1^2}{a_2^3} \nu_{0\cdot0\cdot0\cdot0\cdot0}$$

$$F^* = F_{10} - F'_{10} \, .$$

The last integrations can be obtained immediately, since F_1^* is periodic function of the time t. Thus the results of the first order perturbations show that the elements L_{t0}, G_{t0}, H_{t0}, L_0, G_0, H_0 are periodic functions of the time t and the conjugate elements l_{t0}, g_{t0}, h_{t0}, l_0, g_0, h_0 include further a linear term in the time t beside a periodic part. These results mean that the eccentricity, the semi-major axis of the orbit

T_i $(i = 1, 2)$ and the inclination of the orbital plane to the reference plane $x\,y$ in the first order perturbations are ch anging as a periodic functions of the time t.

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