ON SELECTING \( k \) BALLS FROM AN \( n \)-LINE WITHOUT UNIT SEPARATION

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Let \( l(n, k) \) denote the number of ways of selecting \( k \) balls from \( n \) balls arranged in a line (called an \( n \)-line) with no two adjacent balls, from the \( k \) selected balls, being unit separation. It is shown that \( l(n, k) \) satisfies the recurrence relation

\[
l(n, k) = l(n - 1, k) + l(n - 1, k - 1) - l(n - 2, k - 1) + l(n - 3, k - 1)
\]

and an explicit form for \( l(n, k) \) is obtained.

1. Introduction

Kaplansky\(^1\) proved that,

\[
f(n, k) = \begin{cases} 
\binom{n + 1 - k}{k} & 0 \leq k \leq \frac{n + 1}{2} \\
0 & \text{otherwise},
\end{cases}
\]

where \( f(n, k) \) denotes the number of ways of selecting \( k \) balls from \( n \) balls arranged in a line without any two selected balls being consecutive, (see also Riordan\(^3\), Ryser\(^5\)).

Recently Konvalina\(^2\) proved that

\[
g(n, k) = \begin{cases} 
\sum_{i=0}^{[k/2]} \binom{n + 1 - k - 2i}{k - 2i} & \text{if } n \geq 2(k - 1) \\
0 & \text{if } n < 2(k - 1)
\end{cases}
\]

where \([k/2]\) is the greatest integer less than or equal to \( k/2 \) and \( g(n, k) \) denotes the number of ways of selecting \( k \) balls from \( n \) balls arranged in a line (called an \( n \)-line) without any two selected balls being uni-separate (i.e. being separated by exactly one ball which can be either selected or not).

A related problem is to determine the number of ways of selecting \( k \) balls from an \( n \)-line with no two adjacent balls, from the \( k \) selected balls, being uni-separate. Let \( l(n, k) \) denote the number of ways of selecting \( k \) balls from an \( n \)-line with no two adjacent balls, from the \( k \) selected balls, being uni-separate.
In this paper we derive a recurrence relation for $l(n, k)$ and then give a method similar to that used by Riordan\textsuperscript{4} to determine $l(n, k)$.

2. The Main Result

Lemma 2.1—Let $l(n, 0) = 1$, then $l(n, k)$ satisfies the recurrence relation

$$l(n, k) = l(n - 1, k) + l(n - 1, k - 1) - l(n - 2, k - 1) + l(n - 3, k - 1)$$

...(2.1)

with the boundary conditions $l(n, 1) = n$ and $l(1, k) = 0$ if $k > 1$.

Proof: To prove the recurrence relation (2.1), let $l(n, k)$ be defined as before. Then, the corresponding selections either contain the first ball or they do not. If they do not, then they are enumerated by $l(n - 1, k)$. If they do, then the selections either contain the second ball or they do not. If they do not, then they cannot contain the third ball and, hence, are enumerated by $l(n - 3, k - 1)$. If the selections contain the first and second balls, then the selections either contain the third ball or they do not. If they do not they cannot contain the fourth and, hence are enumerated by $l(n - 4, k - 2)$. If the selections contain the first, the second and the third balls, then the selections either contain the fourth or they do not, and so on. Hence, $l(n, k)$ satisfies the recurrence

$$l(n, k) = l(n - 1, k) + l(n - 3, k - 1) + l(n - 4, k - 2) + \ldots$$

$$= l(n - 1, k) + l(n - 3, k - 1) + L(n, k)$$

where

$$L(n, k) = \sum_{i=2}^{k} l(n - 2 - i, k - i).$$

Then

$$l(n - 1, k - 1) = l(n - 2, k - 1) + l(n - 4, k - 2)$$

$$+ l(n - 5, k - 3) + \ldots$$

$$= l(n - 2, k - 1) + L(n, k)$$

thus the recurrence relation (2.1) is obtained.

Theorem 2.1—

$$l(n, k) = \sum_{i=0}^{\lambda} \binom{k - 1}{i} \binom{n - k + 1 - i}{i + 1} \text{ if } k \leq n$$

where

$$\lambda = \min \left( k - 1, \left\lfloor \frac{n - k}{2} \right\rfloor \right)$$
and \( l(n, k) = 0 \) otherwise.

**Proof:** First we find \( l(n, k) \) for \( k = n, n - 1 \) and \( n - 2 \). It is clear that \( l(n, n) = 1 \), from the boundary conditions and using the recurrence (2.1), we obtain

\[
l(n, n - 1) = l(n - 1, n - 1) + l(n - 1, n - 2) - l(n - 2, n - 2) + l(n - 3, n - 2)
\]

\[
= l(n - 1, n - 2) = \ldots = l(2, 1) = 2
\]

\[
l(n, n - 2) = l(n - 1, n - 2) + l(n - 1, n - 3) - l(n - 2, n - 3) + l(n - 3, n - 3)
\]

\[
= l(n - 1, n - 3) + 1
\]

\[
= l(3, 1) + n - 3
\]

\[
= 3 + n - 3 = n.
\]

It remains to find the explicit form for \( l(n, k) \) if \( k \leq n - 3 \). From the boundary conditions \( l(n, 0) = 1, l(n, 1) = n \), and using the recurrence (2.1) we have

\[
l(n, 2) = l(n - 1, 2) + l(n - 1, 1) - l(n - 2, 1) + l(n - 3, 1)
\]

\[
= l(n - 1, 2) + (n - 2)
\]

\[
= l(2, 2) + (n - 2) + (n - 3) + \ldots + 1
\]

\[
= \frac{(n - 1)(n - 2)}{2} + 1 = \frac{(n-2)(n-3)}{2} + (n-1)
\]

\[
= \binom{n-1}{1} + \binom{n-2}{2}.
\]

Similarly

\[
l(n, 3) = l(n - 1, 3) + l(n - 1, 2) - l(n - 2, 2) + l(n - 3, 2)
\]

\[
= l(n - 1, 3) + \binom{n-3}{2} - \binom{n-4}{2} + \binom{n-5}{2} + (n - 3)
\]

\[
= l(n - 1, 3) + (n - 4) + \binom{n-5}{2} + (n - 3)
\]

\[
= l(n - 1, 3) + \binom{n-4}{2} (n - 2)
\]

which entails

\[
l(n, 3) = \binom{n-2}{1} + 2 \binom{n-3}{2} + \binom{n-4}{3}
\]
By the mathematical induction, we get

\[ l(n, k) = \sum_{i=0}^{\lambda} \binom{k - 1}{i} \binom{n - k + 1 - i}{i + 1} \text{ if } k \leq n \lambda = \min(k - 1, \frac{n - k}{2}). \]

It is clear that \( l(n, k) = 0 \) if \( k > n \), this completes the proof of the theorem.

Finally a table is given for the numbers \( l(n, k) \) where \( n = 0 (1) 12 \).

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