ON SPATIAL NUMERICAL RANGES OF OPERATORS ON BANACH SPACES

C. PUTTAMADAIH AND HUCHE GOWDA

Department of Mathematics, University of Mysore, Manasagangotri
Mysore 570006

(Received 17 November 1986; after revision 16 December 1987)

In this paper a characterization for the spatial numerical range of a normal operator on a smooth reflexive Banach space, to be closed and convex, is given (Theorem 1). This generalizes the Theorem of Ching-Hua-Meng\(^3\), for normal operators on Hilbert spaces. A few more results concerning the spatial numerical ranges of convexoid and isoabelian operators are also obtained.

Let \(X\) be a complex Banach space and \(B(X)\) the Banach algebra of bounded linear operators on \(X\). For \(T \in B(X)\), the spatial numerical range \(V(T)\) and the numerical range \(W(T)\) of \(T\) are given by

\[
V(T) = \{ f(Tx) : x \in X, f \in X^*, \|f\| = \|x\| = f(x) = 1 \}
\]

and

\[
W(T) = \{ [Tx, x] : \|x\| = 1 \}
\]

where \([,\,]\) is a consistent semi-inner-product (s.i.p) of Lumer\(^8\). The sets \(V(T)\) and \(W(T)\) are neither closed nor convex in general. It is easy to see that \(V(T)\) is the union of all the numerical ranges \(W(T)\) corresponding to all consistent s.i.p's on \(X\). However, \(V(T) = W(T)\) in case of smooth space.

\(T \in B(X)\) is said to be hermitian if \(W(T)\) is real or equivalently \(V(T)\) is real. If \(T = H + iK\), where \(H\) and \(K\) are hermitian with \(HK = KH\), then \(T\) is called a normal operator. The numerical range of \(T\) w.r.t the Banach algebra \(B(X)\) is defined by

\[
V(B(X), T) = \{ F(T) : F \in B(X)^*, \|F\| = F(I) = 1 \}.
\]

It is easy to see that \(W(T) \subset V(T) \subset V(B(X), T)\).

An extreme point of a convex set \(S \subset C\) is a point \(\mu\) of \(S\) which does not belong to any line segment joining two points of \(S\) which are different from \(\mu\).

The sets \(\sigma(T), \sigma_c(T), \sigma_c(T), \sigma_r(T), \sigma_r(T)\) and \(\Gamma(T)\) respectively denote the spectrum, convex hull of the spectrum, continuous spectrum, point spectrum, residual
spectrum, approximate point spectrum and the compression spectrum of $T$. An operator $T$ is called convexoid if $\overline{V(T)} = \text{Co} \sigma(T)$. A normal operator on a Banach space $X$ is convexoid. Indeed, if $T$ is a normal operator, then $V(B(X), T) = \text{Co} \sigma(T)$ (Bonsall and Duncan\textsuperscript{1}, Th. 14, p. 54). Since $\text{Co} \sigma(T) \subset \overline{V(T)}$ for any $T \in B(X)$ (Bonsall and Duncan\textsuperscript{2}, Th. 4, p. 22) and $\overline{V(T)} \subset V(B(X), T)$, we have $\overline{V(T)} = \text{Co} \sigma(T)$.

Throughout this this $E$ denotes the set of all extreme points of $\overline{V(T)}$ of a convexoid operator $T$.

**Definition**— A Banach space $X$ is said to be smooth if for each unit vector $x$ in $X$, the set $D(x) = \{f \in X^* : \|f\| = f(x) = 1\}$ contains only one point.

**Theorem 1**—Let $T$ be a normal operator on a smooth reflexive Banach space $X$. Then $V(T)$ is closed and convex if and only if $E \cap \sigma_e(T) = \phi$.

**Proof**: Since $T$ is normal, $\overline{V(T)} = \text{Co} \sigma(T)$, and therefore $E \subset \sigma(T)$. Suppose $E \cap \sigma_e(T) = \phi$. Then as $\sigma_T(T) = \phi$ (Mattila\textsuperscript{9}, Th. 4.7), $E \subset \sigma_p(T)$. Since $\overline{V(T)}$ is a compact convex subset of $C$, we observe that $\overline{V(T)} = \text{Co} E$. As $\text{Co} \sigma_p(T) \subset V(T)$ (Bonsall and Duncan\textsuperscript{2}, Th. 3, p. 21), it follows that $\overline{V(T)} \subset V(T)$. Thus $V(T)$ is closed and convex.

Conversely, suppose $V(T)$ is closed and convex. Since $V(T) \subset V(T^*) \subset \overline{V(T)}$ (Bonsall and Duncan\textsuperscript{2}, Corollary 3, p. 12), $V(T) = V(T^*)$. $T^*$ is normal because $T$ is normal. As $X$ is smooth and reflexive, $X^*$ is strictly convex, and therefore

$$E = E \cap V(T) = E \cap V(T^*) \subset \sigma_p(T^*)$$

(Mattila\textsuperscript{9}, Th. 7.2).

Now the reflexivity of $X$ and $X^*$ implies that $\sigma_e(T) = \sigma_e(T^*)$ (Goldberg\textsuperscript{6}, p. 71) and $\sigma_T(T) = \phi = \sigma_T(T^*)$ (Mattila\textsuperscript{9}, Th. 4.7). Therefore, $\sigma_p(T) = \sigma_p(T^*)$, and hence $E \subset \sigma_p(T)$. Thus $E \cap \sigma_e(T) = \phi$.

We note that Theorem 1 is an extension of the Theorem proved by Ching-Hua-Meng\textsuperscript{3} for normal operators on a Hilbert space.

**Proposition 2**—Let $T$ be a normal operator on a separable reflexive Banach space $X$. If $E \cap \sigma_e(T) = \phi$, then $E$ is countable.

**Proof**: By the first part of Theorem 1, $E \subset \sigma_p(T)$. Since $X$ is separable, $\sigma_p(T)$ $E$ is countable by (Mattila\textsuperscript{9}, Cor. 3.10).

**Proposition 3**—Let $T$ be a hermitian operator on a reflexive Banach space. If $E \cap \sigma_e(T) = \phi$, then there exists an eigenvalue $\alpha$ such that $|\alpha| = ||T||$.

**Proof**: Since $\overline{V(T)} = \text{Co} \sigma(T), E \subset \sigma(T)$. As $T$ is normaloid (Sinclair\textsuperscript{12}, Prop. 2) there is an $\alpha$ in $\sigma(T)$ such that $|\alpha| = ||T||$, and hence $\alpha$ is in $E$. By the first part of Theorem 1, $E \subset \sigma_p(T)$.
Theorem 4—Let $T$ be a convexoid operator on a Banach space. Then

(i) $E \subset \sigma_w(T)$

(ii) If $E \subset \sigma_p(T)$, $V(T)$ is closed and convex.

**Proof:** (i) Since $\overline{V(T)} = Co\,\sigma(T)$, $E \subset \sigma(T)$. As $\sigma(T) \subset \overline{V(T)}$, $E$ cannot be in the interior of $\sigma(T)$. Hence $E \subset \partial\sigma(T) \subset \sigma_w(T)$.

(ii) Since $Co\,\sigma(T) = \overline{\sigma(T)} = Co\,E$ and $Co\,\sigma_p(T) \subset V(T)$ (Bonsall and Duncan², Th. 3), it follows that $V(T)$ is closed and convex.

**Remark 1:** $V(T) \neq V(T^*)$ in general. However, they are equal for a convexoid operator $T$ on a Banach space with $E \subset \sigma_p(T)$. This follows from (ii) of Theorem 4 and Corollary 3 of Bonsall and Duncan² (p. 12).

**Remark 2:** It would be interesting to know whether the converse of (ii) in Theorem 4 is true. However, we prove

Theorem 5—Let $T$ be a normal operator on a strictly convex Banach space. Then a necessary and sufficient condition that $V(T)$ be closed and convex is that $E \subset \sigma_p(T)$.

**Proof:** Since a normal operator is convexoid, the sufficiency follows from (ii) of Theorem 4. Also the condition is necessary for, if $V(T)$ is closed, then

$$E = E \cap V(T) \subset \sigma_p(T)$$

by (Mattila⁹, Th. 7.2).

A non zero vector $x$ in a normed linear space $X$ is said to be orthogonal to $y$ in $X$ w.r.t. a compatible s.i.p. $[,]$ on $X$ if $[y, x] = 0$. This orthogonality is not symmetric in general.

**Lemma 6**—If $M$ is a proper closed subspace of a reflexive Banach space $X$, then there is a non-zero vector $y$ in $X$ and a compatible s.i.p $[,]$ on $X$ such that $y$ is orthogonal to $M$.

This is our Corollary 9 of Puttamađaiah and Huchegowda¹¹.

**Theorem 7**—If $X$ is a reflexive Banach space and $T \in B(X)$, then $\Gamma(T) \subset V(T)$; in particular $\sigma_r(T) \subset V(T)$.

**Proof:** Let $\mu \in \Gamma(T)$. Then $(T - \mu I)x$ is not dense in $X$. Then by Lemma 6, there is a compatible s.i.p $[,]$ on $X$ and a non-zero vector $z$ in $X$ such that $[(T - \mu I)x, z] = 0$ for all $x$ in $X$. In particular, for $x = z$, $\mu = [Tz, z]$ with $\|z\| = 1$ and hence $\mu \in W(T) \subset V(T)$.

Mattila⁹ (Th. 7.3) has proved that if $T$ is a normal operator on a smooth reflexive Banach space, then $E \cap V(T) \subset \sigma_p(T)$; consequently $E - V(T) \subset \sigma_c(T)$ because $\sigma_r(T) = \phi$ (Mattila⁹, Th. 4.7). We now extend the latter result for convexoid operators on a reflexive Banach space.
Theorem 8—If $T$ is a convexoid operator on a reflexive Banach space $X$, then $E - V(T) \subset \sigma_c(T)$.

Proof: Since $\overline{\sigma(T)} = \sigma_c(T)$, $E \subset \sigma(T)$. As $\sigma_p(T) \subset V(T)$ always and $\sigma_c(T) \subset V(T)$ by Theorem 7, it follows that

$E - V(T) \subset \sigma(T) - V(T) \subset \sigma_c(T)$.

For each $x$ in a normed linear space $X$, there is an $x^*$ in $X^*$ such that $x^*(x) = \|x\|^2$ and $\|x^*\| = \|x\|$. Let $\phi$ associate each $x$ in $X$ to exactly one such $x^*$ in $X^*$ and $\alpha x$ to $\alpha x^*$. $\phi$ is called a support mapping. There are infinite number of such mappings unless the space is smooth. Every support mapping $\phi$ defines a compatible s.i.p $[.,.]$ on $X$ if we set $\phi(x)(y) = [y, x]$. An invertible operator $T$ on a Banach space $X$ is said to be iso-abelian if there is a support mapping $\phi$ such that $\phi T = T^{-1} \phi$, or equivalently $(Tx)^* = T^{-1} x^*$ or $[Tx, y] = [x, T^{-1} y]$ for all $x, y$ in $X$ where $[.,.]$ is a consistent s.i.p on $X$ defined by $\phi(x)(y) = [y, x]$.

We note that the following three conditions are equivalent for a bounded linear operator $T$ on a Banach space.

1. $T$ is iso-abelian
2. $T$ is an invertible isometry
3. $T$ is invertible and $\|T\| = \|T^{-1}\| = 1$.

$(1) \Rightarrow (2)$ (Koehler and Rosenthal, Cor. 1). It is easy to see the equivalence of $(2)$ and $(3)$.

Theorem 9—If $X$ is a reflexive Banach space and $T \in B(X)$ with $0 \notin V(T)$, then

(i) $T$ is $1 - 1$

(ii) $\overline{TX} = X$

(iii) If $T$ is an isometry on $X$, then $T$ is iso-abelian.

Proof: (i) Suppose $x$ is a unit vector such that $Tx = 0$. Then by Hahn Banach theorem there is an $f \in X^*$ such that $\|f\| = 1$ and $f(x) = 1$. Now $0 = f(Tx) \in V(T)$, a contradiction. We note that $X$ need not be reflexive in this case.

(ii) Suppose $\overline{TX} \neq X$. Then by Lemma 6, there is a compatible s.i.p $[.,.]$ on $X$ and a non-zero vector $y$ such that $[Tx, y] = 0$ for all $x$ in $X$. In particular, $[Ty, y] = 0$. Since we can take $\|y\| = 1$, $0 \in W(T) \subset V(T)$, this contradiction proves (ii).

(iii) Since $T$ is an isometry, $TX$ is closed, and hence by (ii) $T$ is onto and so it is an invertible isometry. Thus $T$ is iso-abelian.
A point \( \mu \in \partial \sigma (T) \) is said to be a proper boundary point of \( \sigma (T) \) if there exists a bounded sequence \((\mu_n)\) in \( \rho (T) \) such that \( \| (\mu_n - \mu) (\mu_n - T)^{-1} \| \to 1 \) as \( n \to \infty \), where \( \rho (T) \) is the resolvent set of \( T \). The set of all proper boundary points of \( \sigma (T) \) is denoted by \( \rho_r (T) \).

We observe that if \( T \) is iso-abelian, then so is \( T^* \).

**Theorem 10**—If \( T \) is an iso-abelian operator on a reflexive Banach space \( X \), then \( \sigma_r (T) = \phi \).

**Proof** : If \( \mu \in \sigma (T) \), then \( | \mu | = \| T \| \) and so \( \mu \in \partial V (B(X), T) \). Since \( \sigma (T) \cap \partial V (B(X), T) \subset \rho_r (T) \) (Mattila\(^{10}\), Lemma 1), \( \mu \in \rho_r (T) \) or equivalently \( 0 \in \rho_r (T - \mu I) \). Since \( X \) is reflexive, \( X = \text{Ker} (T - \mu I) \oplus (T - \mu I)X \) by (Mattila\(^{9}\), Cor. 4.5). This shows that \( \sigma_r (T) = \phi \).

**Corollary 11**—If \( T \) is an iso-abelian operator on a reflexive Banach space \( X \), then \( \sigma_p (T) = \sigma_p (T^*) \).

**Proof** : Since \( T^* \) is iso-abelian, the reflexivity of \( X \) and \( X^* \) implies that \( \sigma_c (T) = \sigma_c (T^*) \) (Goldberg\(^{3}\), p. 71) and \( \sigma_r (T) = \phi = \sigma_r (T^*) \) by Theorem 10.

**Theorem 12**—If \( T \) is an iso-abelian operator on a smooth reflexive Banach space \( X \), then \( \sigma (T) \cap V (T) \subset \sigma_p (T) \).

**Proof** : It is clear that \( \sigma (T^*) \) lies on the unit circle. Since \( X \) is smooth and reflexive, \( X^* \) is strictly convex, and therefore by (Bonsall and Duncan\(^{1}\), Th. 8, p. 93) \( \sigma (T^*) \cap V (T^*) \subset \sigma_p (T^*) \). Since \( V (T) \subset V (T^*) \) and \( \sigma (T) = \sigma (T^*) \), \( \sigma (T) \cap V (T) \subset \sigma (T^*) \cap V (T^*) \). The proof is completed by Corollary 11.

**Corollary 13**—If \( T \) is an iso-abelian operator on a smooth reflexive Banach space \( X \), then the extreme points of \( Co \sigma (T) \) which lie in \( V (T) \) are the eigen values of \( T \).

**Theorem 14**—If \( T \) is a normal and iso-abelian operator on a smooth reflexive space \( X \), then \( V (T) \) is closed if and only if the spectrum of \( T \) consists entirely of the point spectrum.

**Proof** : Since \( T \) is an invertible isometry, \( \sigma (T) \) lies on the unit circle. As \( T \) is normal, \( V (T) = Co \sigma (T) \) and therefore it follows that \( E = \sigma (T) \). Suppose \( V (T) \) is closed. Then by Theorem 1, \( E \cap \sigma_c (T) = \phi \). But this holds only if \( \sigma (T) = \sigma_p (T) \) because \( \sigma_r (T) = \phi \), by Theorem 10.

The converse is an immediate consequence of the fact that \( T \) is convexoid and \( Co \sigma_p (T) \subset V (T) \) (Bonsall and Duncan\(^{8}\), Th. 3, p. 21).

**Theorem 15**—If \( T \) is an iso-abelian (or normal) operator on a Banach space with descent \( \partial (T - \mu I) \) finite for each \( \mu \) in \( \sigma (T) \) then \( Co \sigma (T) \subset V (T) \).

**Proof** : By Corollary 3.6 of Mattila\(^{8}\), the ascent \( \alpha (T - \mu I) \leq 1 \) for each \( \mu \) in \( C \) and hence \( \partial (T - \mu I) = \alpha (T - \mu I) \leq 1 \) for each \( \mu \) in \( \sigma (T) \). Now \( \mu \) is an isolated
point of $\sigma(T)$ and is a pole of the resolvent of $T$ (Lay, Th. 2.1). Therefore $\mu$ is an eigen value of $T$ (Taylor, Th. 5.8 - A). Hence $\sigma(T) = \sigma_p(T)$. Since $C\sigma_p(T) \subset V(T)$ (Bonsall and Duncan, Th. 3), the proof is completed.

**Corollary 16**—Under the hypothesis of the Theorem, $\sigma(T) \subset W(T)$.

**Corollary 17**—If $T$ is a normal operator on a Banach space with $\delta(T - \mu I)$ finite for each $\mu$ in $\sigma(T)$, then $V(T)$ is closed and convex.

**Acknowledgement**

The authors wish to express their heartfelt thanks to the Referee for his valuable suggestion.

**References**