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SCATTERING OF COMPRSSIONAL WAVES BY A CIRCULAR CYLINDER

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This paper deals with the scattering of compressional waves by a circular cylinder. The cylinder is embedded in an unbounded isotropic homogeneous elastic medium and it is filled with some acoustic fluid.T he line source, generating the incident pulse is situated outside the cylinder parallel to its axis. The problem is investigated by the method of dual integral transformations. The resulting integrals are evaluated asymptotically to obtain the short time estimate of the motion near the wave front in the illuminated region of the elastic medium. We interpret the approximate solution in terms of geometrical optics.

1. INTRODUCTION

Lapwood[4] considered the problem on the disturbance due to a line source in a semi-infinite elastic medium and obtained the exact formal solution of the problem. Jeffreys and Lapwood[10] discussed the reflection of a pulse within a sphere and obtained the solution using operational methods. The problem on scattering of two dimensional elastic waves with a cylindrical obstacle in unbounded medium has been considered by various autors in recent years. Gilbert and Knopoff[7] used Friedlander's[5] method to investigate the problem of scattering of impulsive elastic waves by a rigid circular cylinder. Gilbert[6] discussed the problem of scattering of impulsive elastic waves by a smooth convex cylinder using the same method. Mishra[15,16] applied Friedlander's method to investigate the problem of scattering and diffraction of two dimensional sound pulses by a acoustically semi-transparent circular cylinder. Hwang *et al.*[13] applied a similar method and discussed the case of three dimensional elastic waves scattring by a rigid cylinder in an elastic medium.

In this paper, we investigate the short-time approximation for the scattering of compressional waves by a circular cylinder filled with inviscid fluid material. The cylinder is supposed to be situated in an unbounded homogeneous isotropic elastic medium and the incident pulse is generated by a line source situated in the surrounding elastic medium at a finite distance parallel to the axis of cylinder. We assume that the velocities of P and SV waves outside the cylinder are α and β respectively and that of P-waves inside the cylinder is α_0. To be specific, we suppose $\alpha > \alpha_0 > \beta$. This assumption of the velocity distribution corresponds to the actual velocity distribution of elastic waves inside the earth and to the location of the source in the mantle and the outer core as the obstacle[12]. We also suppose the density of the medium outside the cylinder is ρ and that inside the cylinder is ρ' where $\rho > \rho'$. The present discussion therefore may have some relevance in seismological problems.

2. Formulation of the Problem and Formal Solution

Let the axis of the cylinder be taken as the z-axis and let a co-ordinate (r, θ) be located in the (x, y) plane with $\theta = 0$, $r = r_0$, $(r_0 > a)$ corresponding to the location of the line source which is parallel to the axis of the cylinder. The equation of the cylinder is $r = a$.

The governing equations for the present case are

$$\frac{1}{\alpha^2}\frac{\partial^2 \varphi}{\partial t^2} - \nabla^2\varphi = \frac{2\pi}{r}\,\delta(r - r_0)\,\delta(t)\,\delta(\theta),\ (r \geqslant a) \qquad \ldots(2.1)$$

$$\frac{1}{\beta^2}\frac{\partial^2 \psi}{\partial t^2} - \nabla^2\psi = 0 \qquad (r \geqslant a) \qquad \ldots(2.2)$$

$$\frac{1}{\alpha_0^2}\frac{\partial^2 \varphi_0}{\partial t^2} - \nabla^2\varphi_0 = 0, \qquad (r \leqslant a) \qquad \ldots(2.3)$$

where, ∇^2 is the Laplacian operator.

Besides the potentials also satisfy the conditions of quiescence at $t = 0$. The boundary conditions for the problem are that the tangential and normal components of stress vanish outside the obstacle and the radial displacement is continuous at the boundary of the obstacle[17].

Applying Laplace and Fourier transformations[5] to (2.1), (2.2) and (2.3), and following Mishra[15,16], we find that Laplace transforms of the solutions are given by

$$\bar{\varphi}(r, \theta, s) = \int_{-\infty}^{\infty} I_\mu\left(\frac{sr}{\alpha}\right) K_\mu\left(\frac{sr_0}{\alpha}\right) \exp(i\mu\theta)\, d\mu$$

$$+ \int_{-\infty}^{\infty} K_\mu\left(\frac{sr}{\alpha}\right) K_\mu\left(\frac{sr_0}{\alpha}\right) \frac{L}{M} \exp(i\mu\theta)\, d\mu,\ (r_0 \geqslant r > a) \qquad \ldots(2.4)$$

$$\bar{\psi}(r, \theta, s) = \int_{-\infty}^{\infty} K_\mu\left(\frac{sr}{\beta}\right) K_\mu\left(\frac{sr_0}{\alpha}\right) \frac{N}{M} \exp(i\mu\theta)\, d\mu,\ (r \geqslant a) \qquad \ldots(2.5)$$

and

$$\bar{\varphi}_0(r, \theta, s) = \int_{-\infty}^{\infty} \frac{L.R + N.P}{M.Q}\, I_\mu\left(\frac{sr}{\alpha_0}\right) K_\mu\left(\frac{sr_0}{\alpha}\right) \exp(i\theta\mu)\, d\mu,\ (r \leqslant a). \qquad \ldots(2.6)$$

Here L, M, N, P, Q and R are the values of the constants which are determined with the help of boundary conditions.

We see that (2.4), (2.5) and (2.6) give the integral representation of Laplace transform of the formal solutions. The time solution can be obtained on performing Laplace inversion.

3. Incident, Reflected and Refracted Pulses

We first give a brief description of the geometry of the problem. Initially the incident P-pulse striking the outer surface of the cylinder gives rise to reflected P, reflected S and refracted P-pulses, acording to the laws of ordinary geometrical optics (Fig. 1) when the rays strike the outer surface of the cylinder at critical angle, the reflected P-rays become tangential to the surface and as such they move along the surface. These surface waves at each point of their path give rise to SV-wave at

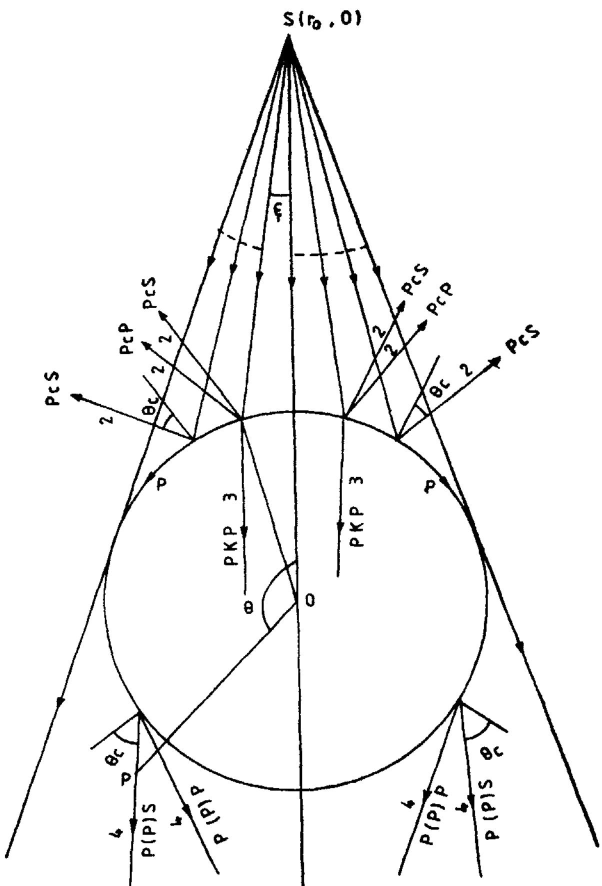

Fig. 1. (1) Incident rays, (2) reflected rays, (3) refracted rays and (4) diffracted rays.

critical angle in the outer medium and P-wave along the tangent to the surface[11]. The former are denoted by $P\,(P)\,S$ and latter by $P\,(P)\,P$ (Gilbert and Knopoff[7]). In the case of grazing incidence, the disturbances move along the surface and at each point of their path they shed diffracted P-waves in the outer medium trangential to the surface. These are denoted by $P\,(P)\,P$ (Bullen[12]). The reflected pulses are denoted by $P_c\,P$ and P_cS and refracted pulse is denoted by PKP (Bullen[12]).

Now, we use the saddle point method[9] to obtain the solution in the illuminated region of the elastic medium. Therefore using the various approximations for modified Bessel functions as given by Mishra[15,16] to the integral (2.4), (2.5) and (2.6) we find

$$\bar{\varphi}\,(r,\theta,s) \backsim \left(\frac{\alpha\pi}{2\,sR_1}\right)^{1/2} \exp(-st_1) + \left(\frac{\pi}{s}\right)^{1/2} A_1\,B_1 \exp(-st_2) \qquad (r_0 \geqslant r \geqslant a) \qquad \ldots(3.1)$$

$$\bar{\psi}\,(r,\theta,s) \backsim \left(\frac{\pi}{s}\right)^{1/2} A_2\,B_2 \exp(-st_3), \qquad (r \geqslant a) \qquad \ldots(3.2)$$

and

$$\bar{\varphi}_0\,(r,\theta,s) \backsim \left(\frac{\pi}{s}\right)^{1/2} A_3\,B_3 \exp(-st_4), \qquad (r \leqslant a) \qquad \ldots(3.3)$$

where, t_1, t_2, t_3 and t_4 are respectively the arrival times of the incident, reflected and refracted pulses.

$$A_1 = \frac{\frac{4\rho\beta^3}{\alpha^2}\sin^2\eta\cos\eta\,(1-n_1^2\sin^2 n)^{1/2}(1-n^2\sin^2\eta)^{1/2} + \alpha_0\,\rho'\cos\eta - \rho\alpha\,(1-2n^2\sin^2\eta)^2\,(1-n_1^2\sin^2\eta)^{1/2}}{4\rho\,\beta^2\alpha^{-2}\sin^2\eta\cos\eta\,(1-n_1^2\sin^2 n)^{1/2}\,(1-n^2\sin^2\eta)^{1/2} + \rho'\,\alpha_0\cos\eta + \rho\alpha\,(1-2\,n^2\sin^2\eta)^2\,(1-n_1^2\sin^2 n)^{1/2}}$$

$$B_1 = \left[\frac{a\alpha\cos\eta}{r_0\,R_3\cos\xi + r\,R_2\cos\zeta}\right]^{1/2}$$

A_2, B_2 and A_3, B_3 have similar expressions as above.

R_1, R_2, R_3 denote distances between the source and receiver and the angles are defined in Figs. 2, 3, 4 and 5.

Now we can obtain the short-time approximations for the solution by performing Laplace inversion[14]. We find that

$$\varphi\,(r,\theta,t) \backsim \frac{H(t-t_1)}{2t_1\,(t-t_1)^{1/2}} + A_1\,B_1\,\frac{H(t-t_2)}{(t-t_2)^{1/2}} \quad (r_0 \geqslant r \geqslant a) \qquad \ldots(3.4)$$

$$\psi\,(r,\theta,t) \backsim A_2\,B_2\,\frac{H(t-t_3)}{(t-t_3)^{1/2}}, \quad (r \geqslant a) \qquad \ldots(3.5)$$

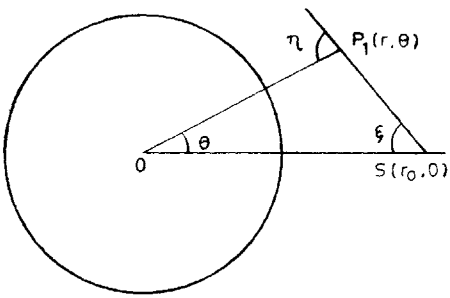

FIG. 2. Geometrical interpretation of the saddle point for the incident P pulse.

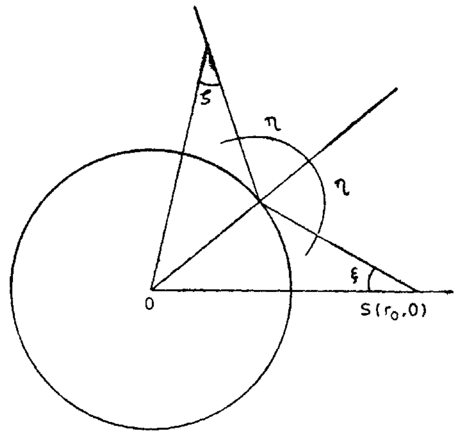

FIG. 3. Geometrical interpretation of the saddle point for the reflected P pulse.

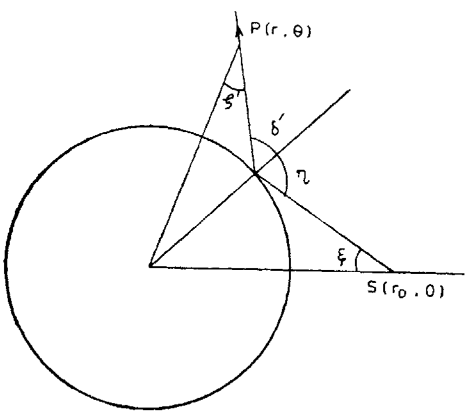

FIG. 4. Geometrical interpretation of the saddle point for the reflected S pulse.

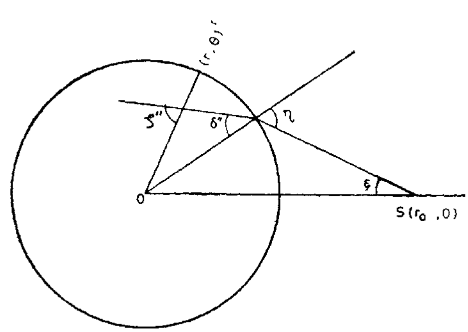

FIG. 5. Geometrical interpretation of the saddle point for the refracted P pulse.

$$\varphi_0(r, \theta, t) \propto A_3 B_3 \frac{H(t - t_4)}{(t - t_4)^{1/2}}, \ (r \leqslant a) \quad ...(3.6)$$

Figures 6-8 present the numerical evaluation of these pulses respectively.

COMPARISON

Hwang *et al.*[13] discussed the problem of three dimensional elastic wave scattering due to a rigid cylinder in case of a compressional point source on comparing our results with those obtained by them, we find that in addition to the results obtained by us, they obtain an additional event in the illuminated region which they term as

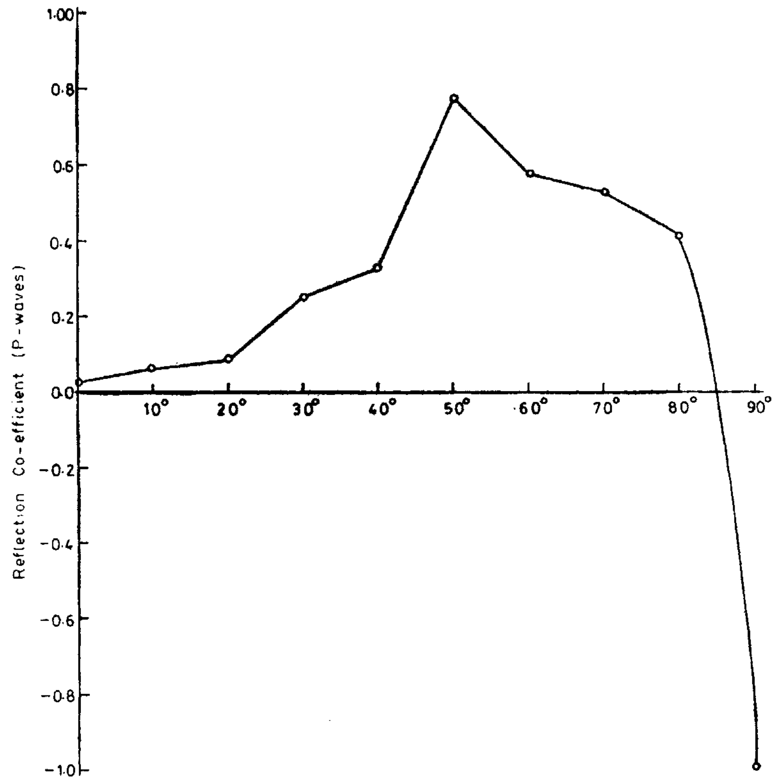

FIG. 6. Reflection co-efficients for P waves.

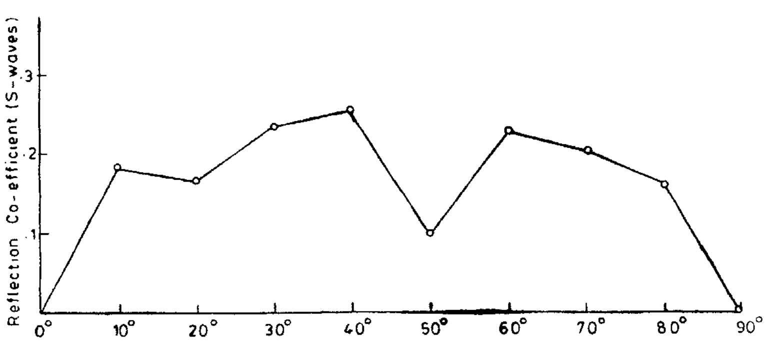

FIG. 7. Reflection co-efficients for S waves.

diffracted PP_dS wave. It may be pointed out that this possibility does not hold in the illuminated region. The physical ground for this is that diffracted waves exist in the shadow region only. Diffracted field is in fact the contribution of poles. It is obtained on evaluating the integrals (2.4) and (2.5) by Watsons residue method. Besides the results obtained by the Saddlepoint method are interpretable only in the

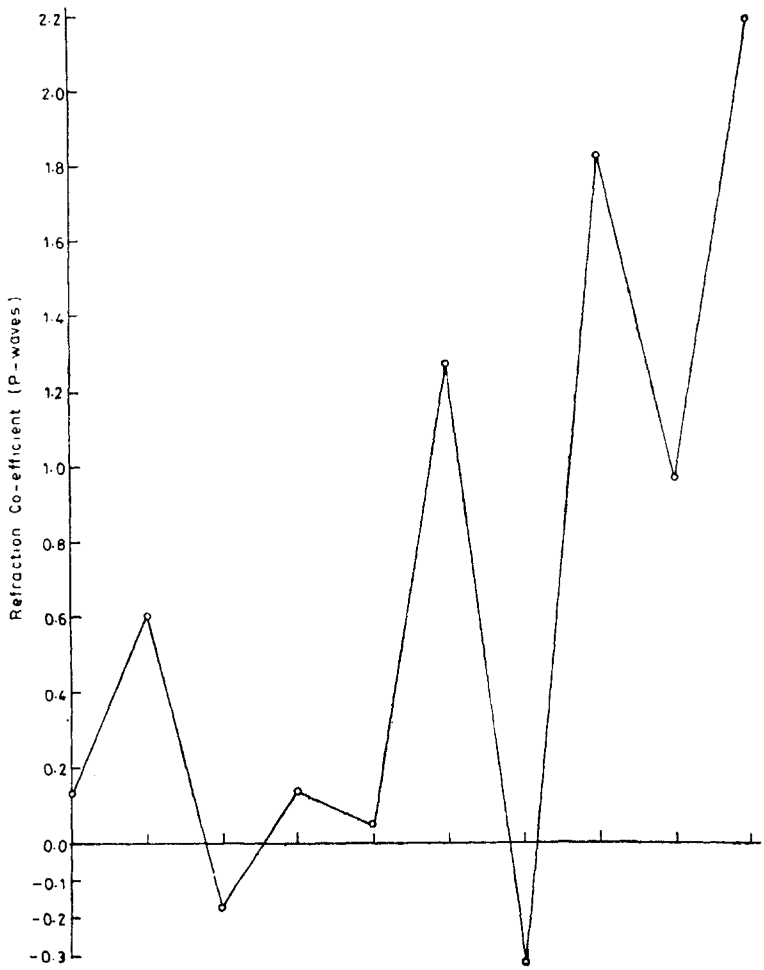

Fig. 8. Transmission co-efficient for *P* waves.

illuminated region[1]. This conclusion seems quite reasonable. It agrees with the results obtained by Mishra[15,16] and Rajhans and Agarwal[3].

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