THREE-DIMENSIONAL MAGNETOFLUIDDYNAMIC FLOW WITH PRESSURE GRADIENT AND FLUID INJECTION

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The non-linear partial differential equations of motion for three-dimensional flow of an incompressible viscous fluid flowing over a semi-infinite plate under the influence of a magnetic field and a pressure gradient, and with or without suction or injection through the plate wall are developed with necessary boundary conditions. The group-theoretic methods are employed to obtain proper similarity transformations. Similarity solution of such flow system yields coupled non-linear ordinary differential equations.

1. INTRODUCTION

Due to increasing number of technical applications using magnetohydrodynamic effect, it is desirable to extend many of the available viscous hydromagnetic solutions to include the effects of magnetic fields for those cases when the viscous fluid is electrically conducting. Flow past a flat plate has been studied in Rossw¹ and Carrier and Greenspan². Heat transfer for these cases has been discussed in Rossw¹ and Afzal³. Rossw¹ has considered transverse magnetic field whereas Carrier and Greenspan² have studied the effect of a longitudinal magnetic field on the velocity and the temperature distributions. There have been other computations of viscous magnetohydrodynamic flow reported in the literature⁴⁵. Djuic⁶ has considered only Hiemenz magnetic flow of non-Newtonian power-law fluid. He has solved the governing non-linear ordinary differential equations using Galerkin's technique, whereas Srivastava and Usha⁷ has discussed magnetofluidynamics of simple two-dimensional flow with pressure gradient and fluid injection. Recently such case is extended by Timol⁸ for a non-Newtonian power-law fluid.

2. BASIC EQUATIONS

The basic equations of magnetofluiddynamics and conventional fluiddynamics are different by only additional force term due to electromagnetic field in momentum equation and a term due to Joule heating in the energy equation. In such situation
the Maxwell's equations have to be satisfied in the entire flow field as well as in the body and interface.

In order to derive the basic equations, the following assumptions are made:

1. The fluid under consideration is incompressible finitely conducting with constant physical properties.
2. Hall effect, electrical and polarization effects are neglected.
3. The induced magnetic field is neglected.
4. The flow is steady and laminar and the imposed magnetic field is perpendicular to the free stream velocities.
5. The magnetic Reynolds number is assumed to be small.

Under these assumptions we now write continuity and momentum equations governing the velocity distribution in the presence of magnetic field as,

\[ \nabla \cdot \vec{v} = 0 \] ... (1)

\[ \rho \vec{v} \cdot \nabla \vec{v} = - \nabla P + \rho \gamma_H \nabla^2 \vec{v} + \vec{J} \times \vec{B} \] ... (2)

where the third term on the right-hand side of eqn. (2) is the Lorentz force due to the magnetic \( \vec{B} \), and is given by

\[ \vec{J} \times \vec{B} = \sigma (\vec{V} \times \vec{B}) \times \vec{B}. \] ... (3)

We now consider the flow past a semi-infinite flat plate placed in the direction of the flow. The plate is in \( X - Z \) plane and is between \( 0 \leq x < \infty \) and \(-\infty < Z < \infty \) and free stream is in the \( X \)-direction as shown in Fig. 1. Here we shall consider that all flow quantities are independent of the \( Z \)-coordinate. Such flows are characterized by the fact that their stream lines form a system of 'translates'. That is stream line pattern can be obtained by translating any particular stream line parallel to the leading edge of the surface. It is hoped that by assuming independence of flow quantities in one direction, more quantitative information may be obtained on the characteristics of three-dimensional boundary layer flows.

Thus the problem considered here is essentially a quasi-two-dimensional one. The magnetic field vector \( \vec{B} \) is perpendicular to the free stream and is along \( Y \)-direction because the induced magnetic field is neglected. The basic equations of the steady, viscous, laminar, three-dimensional boundary layer flow of a fluid flowing over a semi-infinite flat plate under the influence of transverse magnetic field and pressure gradient with or without suction or injection through the plate wall can be derived from the eqns. (1) and (2) on the basis of usual boundary layer assumptions, as

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \] ... (4)
Fig. 1. Flow over a plate in rectangular coordinate system under the influence of transverse magnetic field.

\[ \frac{u}{\partial x} + \frac{v}{\partial y} = - \frac{g}{\rho} \frac{dp}{dx} + \gamma_H \frac{\partial^2 u}{\partial y^2} - [g \sigma B_z^2 (x) \rho] u \]  \hspace{1cm} \ldots (5)

\[ \frac{u}{\partial x} + \frac{v}{\partial y} = - \frac{g}{\rho} \frac{dp}{dz} + \gamma_H \frac{\partial^2 w}{\partial y^2} - [g \sigma B_z^2 (x) \rho] w. \]  \hspace{1cm} \ldots (6)

The boundary conditions for such flow system are taken as,

**Case I**

\[ y = 0 \Rightarrow u = 0; \quad v = V_0 x^{(m-1)/2}; \quad w = 0 \]  \hspace{1cm} \ldots (7)

\[ y \to \infty \Rightarrow u \to U_0 x^m; \quad w \to W (x) = W x^m. \]  \hspace{1cm} \ldots (8)

**Case II**

\[ y = 0 \Rightarrow u = 0; \quad v = V_0 \exp \left( \frac{mx}{2} \right); \quad w = 0 \]  \hspace{1cm} \ldots (9)

\[ y \to \infty \Rightarrow u \to U (x) = U_0 \exp (mx) \]

\[ w \to W (x) = W_0 \exp (mx). \]  \hspace{1cm} \ldots (10)

where \( x \) and \( z \) are the coordinates parallel to the plate, \( y \) a distance from the plate; \( u, v, w \) the velocity components of the fluid along \( x, y \) and \( z \) directions respectively; \( U, W \) velocity components in the main flow along \( x \) and \( z \) direction respectively; \( g \) the acceleration of the gravity; \( P \) the pressure; \( \gamma_H \) the magnetic viscosity of the fluid \( (= m/\rho) \) \( \rho \)—the fluid density; \( m \) a physical constant, \( \sigma \) (Greek 'sigma') the electrical
conductivity of the fluid; \( B_r \) the imposed magnetic induction parallel to \( y \) axis; \( U_0, V_0, W_0, m > 0 \) are constants.

Now introduce stream function \( \psi (x, y) \) such that

\[
u = - \frac{\partial \psi}{\partial x}, \quad v = - \frac{\partial \psi}{\partial y}.
\]

...(11)

Then, the equation of continuity (4) gets satisfied identically and equations of motion (5) and (6) with boundary conditions (7)-(10) will be,

\[
\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} + G_1 (x) + \gamma_H \frac{\partial^2 \psi}{\partial y^2} - H_1 (x) \frac{\partial \psi}{\partial y} = 0
\]

...(12)

\[
\frac{\partial \psi}{\partial y} \frac{\partial w}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial w}{\partial y} = G_2 (x) + \gamma_H \frac{\partial^2 w}{\partial y^2} - H_1 (x) w
\]

...(13)

where

\[
G_1 (x) = - \frac{g}{\rho} \frac{dp}{dx}
\]

...(14)

\[
G_2 (x) = - \frac{g}{\rho} \frac{dp}{dz}
\]

...(15)

\[
H_1 (x) = \frac{g a B_r^2 (x)}{\rho}
\]

...(16)

Then the boundary conditions are,

**Case I**

\[
y = 0 \Rightarrow \frac{\partial \psi}{\partial y} = 0; \quad \frac{\partial \psi}{\partial x} = - V_0 x^{(m-1)/2}; \quad w = 0
\]

...(17)

\[
y \to \infty \Rightarrow \frac{\partial \psi}{\partial y} \to U (x) = U_0 x^m; \quad w \to W (x) = W_0 x^m.
\]

...(18)

**Case II**

\[
y = 0 \Rightarrow \frac{\partial \psi}{\partial y} = 0, \quad \frac{\partial \psi}{\partial x} = - V_0 \exp \left(\frac{m - 1}{2}\right); \quad w = 0
\]

...(19)

\[
y \to \infty \Rightarrow \frac{\partial \psi}{\partial y} \to U (x) = U_0 \exp (mx); \quad w \to W (x) = W_0 \exp (mx).
\]

...(20)

The goal of reducing partial differential equations (12) and (13) to ordinary differential equations with the meaningful boundary conditions (17) - (20) by similarity technique will cause restrictions to be imposed on the functions \( G_1 (x), G_2 (x) \) and \( H_1 (x) \). In order to obtain suitable similarity variables, the group theoretic method is employed.
3. Group-Theoretic Analysis

Similarity analysis by the group-theoretic technique is based on the concept derived from the theory of transformations of groups. Recently this technique is found to give adequate treatment of boundary layer equations. The basic concept of this method was first introduced by Birkhoff\textsuperscript{9} and later on it was extended exclusively by Hansen\textsuperscript{10}, Morgan\textsuperscript{11}, Ames\textsuperscript{12} and others. For the present problem we shall employ two distinct classes of one parameter groups of transformation—a linear group of transformation and a spiral group of transformation and also we shall discuss both the cases separately.

**Case I**

A one parameter group of transformation $\Gamma_1$ is selected as,

$$
\Gamma_1 = \begin{cases} 
  x = A^{a_1} \bar{x} \\
  y = A^{a_2} \bar{y} \\
  \psi = A^{a_3} \phi \\
  w = A^{a_4} \bar{w}
\end{cases}
$$

where $a_1, a_2, a_3, a_4$ and $A$ are real arbitrary constants.

We now seek relations among $\alpha$'s such that the basic equations (12), (13) along with the boundary conditions (17) and (18) will be invariant under this group of transformation $\Gamma_1$. This suggests that $G_1 (x)$, $G_2 (x)$ and $H_1 (x)$ are to be selected as

$$
G_1 (x) = g_0 x^{a_0} 
$$

$$
H_1 (x) = h_0 x^{a_6} 
$$

$$
G_2 (x) = g_1 x^{a_7}. 
$$

Now invariant conditions demand that the power of $A$ in each term of the transformed equations should be equal, which yields

$$
2a_3 - 2a_2 - a_1 = a_6 a_1 = a_3 - 3 a_2 = a_6 a_1 + a_3 - a_2 
$$

$$
a_4 + a_3 - a_2 - a_1 = a_7 a_1 = a_4 - 2a_2 = a_6 a_1 + a_4. 
$$

Further, from the boundary conditions (17), (18), we get,

$$
\frac{a_3}{a_1} = \frac{m + 1}{2}; \frac{a_2}{a_1} = \frac{1 - m}{2}; \frac{a_4}{a_1} = m
$$

and from (24), (25) and (26), we get,

$$
a_6 = 2m - 1; a_4 = m - 1; a_7 = 2m - 1.
$$

Now invariants of the group $\Gamma_1$ are,

$$
ay^{x^{a_1}}, bx^{a_2}/a_1, \text{ and } wcx^{a_4}/a_1
$$

where $a$, $b$ and $c$ are constants. This enable us to derive the following similarity independent and dependent variables:
\[ \eta = ay/x^{(m-1)/2} \]  
(28)

\[ f_1(\eta) = \psi/b \cdot x^{(m+1)/2} \]  
(29)

\[ f_2(\eta) = w/c \cdot x^m. \]  
(30)

Substituting eqns. (28), (29) and (30) in eqns. (12) and (13) and simplifying, we get,

\[ \lambda_1 f_1'' - f_1' f_1 = \frac{\gamma H a_0^2}{\lambda_2 U_0} f_1' + \frac{g_0}{\lambda_2 U_0^2} - \frac{h_0}{\lambda_2 U_0} f_1 \]  
(31)

\[ \lambda_1 f_1' f_2 - f_1 f_2' = \frac{H a_0^2}{\lambda_2 U_0} f_2' + \frac{g_1}{\lambda_2 U_0 W_0} - \frac{h_0}{\lambda_2 U_0} f_2. \]  
(32)

where

\[ \lambda_1 = \frac{2m}{m+1}, \lambda_2 = \frac{m+1}{2} \]  
(33)

\[ G_1(x) = g_0 x^{5m-1} \]  
(34)

\[ G_2(x) = g_1 x^{2m-1} \]  
(35)

\[ H_1(x) = h_0 x^{m-1} \]  
(36)

and a prime denotes differentiation with respect to \( \eta \).

Also from the boundary condition (18), it follows that

\[ U_0 = ab; \quad W_0 = c. \]  
(37)

Setting

\[ S_0 = \frac{g_0}{\lambda_2 U_0^2}, S_1 = \frac{g_1}{\lambda_2 U_0 W_0} \quad \text{and} \quad \mu_0 = \frac{h_0}{\lambda_2 U_0} \]  
(38)

and

\[ \frac{\gamma H a_0^2}{\lambda_2 U_0} = 1 \]  
(39)

eqns. (31) and (32) will become

\[ f_1'' + f_1 f_1' - \lambda_1 f_1' + S_0 - \mu_0 f_1' = 0 \]  
(40)

\[ f_2' + f_1 f_2' - \lambda_1 f_1' f_2 + S_1 - \mu_0 f_2 = 0. \]  
(41)

Now when free stream velocities are function of \( x \), then pressure gradients will be given by:

\[ -\left(\frac{g}{\rho}\right) \frac{dp}{dx} = U \frac{dU}{dx} + \frac{g \sigma B_0^2 (x)}{\rho} U + \partial U/\partial t \]  
(42)
\[- (g/\rho) \frac{dp}{dz} = U \frac{dW}{dx} + \frac{g\alpha B^2 (x)}{\rho} W + \partial W / \partial t. \quad (43)\]

But for steady flow, free stream velocities outside the boundary layer is independent of time i.e.
\[\frac{\partial U}{\partial t} = 0; \quad \frac{\partial W}{\partial t} = 0 \quad (44)\]

under (14)—(16), (17) and (34)—(35), the relationship of (42)—(43) will be,
\[S_0 = \lambda_1 + \mu_0 \quad (45)\]
\[S_1 = \lambda_1 + \mu_0 \quad (46)\]

Substituting (45) and (46) in the equations (40) and (41) respectively, we get,
\[f_1'' + f_1 f_1' + \lambda_1 (1 - f_1^{'2}) + \mu_0 (1 - f_1') = 0 \quad (47)\]
\[f_2'' + f_1 f_2' + \lambda_1 (1 - f_2' f_2) + \mu_0 (1 - f_2) = 0. \quad (48)\]

Equations (47), (48) are coupled non-linear ordinary differential equations for the function of \(f (\eta)\). With the transformed boundary conditions,
\[\eta = 0 \Rightarrow f_1' = -\frac{V_o}{b\lambda_2}, f_2 = 0 \quad (49)\]
\[\eta \to \infty \Rightarrow f_1' = 1, f_2 = 1. \quad (50)\]

From equations (37) and (39) values of \(a, b\) and \(c\) will be
\[a = \left(\frac{\lambda_2 U_0}{\gamma_H}\right)^{1/2}; b = \left(\frac{\gamma_H U_0}{\lambda_2}\right)^{1/2}; c = W_0. \quad (51)\]

Case II

One parameter spiral group of transformation \(\Gamma_2\) can be chosen in the form,
\[\Gamma_2 = \left\{ x = \beta_1 b + \bar{x}, y = e^{B_2 b} y \right\}\]
\[\psi = e^{B_2 b} \bar{\psi}, w = e^{B_4 b} \bar{w} \]

where \(\beta_1, \beta_2, \beta_3, \beta_4\) and \(b\) are real constants and \(e\) is a real parameter of the group transformation \(\Gamma_2\).

In this case \(\Gamma_2\) enables us that \(G_1 (x), G_2 (x)\) and \(H_1 (x)\) are to be selected in such a way,
\[G_1 (x) = g_0 e^{B_2 x} \quad (52)\]
\[G_2 (x) = g_1 e^{B_3 x} \quad (53)\]
\[H_1 (x) = h_0 e^{B_4 x} \quad (54)\]
following the same procedure as in case-I, the following absolute invariants are obtained:

\[
\xi = a_1 \exp \left( \frac{1}{2} m x \right) \quad \text{(55)}
\]

\[
\psi = b_1 \exp (mx) F_1 (\xi) \quad \text{(56)}
\]

\[
w = c_1 \exp (mx) F_2 (\xi) \quad \text{(57)}
\]

where

\[
a_1 = \left( \frac{m U_0}{2 \gamma H} \right)^{1/2}, \quad b_1 = \left( \frac{2 \gamma H U_0}{m} \right)^{1/2}, \quad c_1 = W_0. \quad \text{(58)}
\]

Under (55)—(57), finally eqns. (12)—(13) with the boundary conditions (18), (19) will be transformed to following non-linear ordinary differential equation:

\[
F_1'^{*} + F_1 F_1^{*} + 2 (1 - F_1^{*}) + \mu_0 (1 - F_1') = 0 \quad \text{(59)}
\]

\[
F_2'^{*} + F_1 F_2^{*} + 2 (1 - F_1' F_2) + \mu_0 (1 - F_2) = 0 \quad \text{(60)}
\]

where \( \mu_0 = \frac{2h_o}{mU_o} \) and prime denotes differentiation with respect to \( \xi \)

with the boundary conditions

\[
\xi = 0; \quad F_1' = \frac{-2V_0}{(m \gamma H U_0)^{1/2}}; \quad F_2 = 0 \quad \text{(61)}
\]

\[
\xi = \infty; \quad F_1' = 1; \quad F_2 = 1. \quad \text{(62)}
\]

Equations (59), (60) with the boundary conditions (61), (62) constitute a pair of non-linear ordinary coupled differential equations.

4. Conclusion

The analysis of laminar, incompressible, three-dimensional magneto-fluiddynamic boundary layer equations of viscous fluids with stream lines forming a system of 'translates' lead to the following conclusions:

(i) If we take \( V_o = 0 \) in eqns. (59) and (61), we have magneto-fluiddynamic boundary layer problem without suction or injection through the plate wall which either in Case-I or in Case-II is reduced to a solution of a boundary value problem of following third-order non-linear coupled ordinary differential equations:

\[
f_1'^{*} + f_1 f_1^{*} + \lambda_1 (1 - f_1^{*}) + \mu_0 (1 - f_1') = 0 \quad \text{(63)}
\]

\[
f_2'^{*} + f_1 f_2^{*} + \lambda_1 (1 - f_1' f_2) + \mu_0 (1 - f_2) = 0 \quad \text{(64)}
\]
where

\[ \lambda_1 > 0; \mu_0 > 0. \]

The boundary conditions are

\[ f_1(0) = 0; f'_1(0) = 0; f_2(0) = 0 \] \hspace{1cm} \text{(65)}

\[ f'_1(\infty) = 1, f_2(\infty) = 1. \] \hspace{1cm} \text{(66)}

(ii) For the non-magnetic case i.e. for \( \mu_0 = 0 \), the above set of eqns. (63)–(64) will reduce to the set of equations for simple three-dimensional viscous incompressible flows obtained by Timol et al.\(^{19} \) (for power index \( n = 1 \) and \( \lambda_1 = 1/3 \)).

(iii) For \( m = 1, U(x) = U_0 x, W(x) = W_0 x \), the entire problem will reduce to the particular case of three-dimensional magnetic flow near a stagnation point i.e. Hiemenz three-dimensional magnetic flows. In such situation the transverse magnetic field will be found constant through the plate wall. The present problem is recently extended to include non-Newtonian fluids of different models by Timol\(^{14} \).

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References