A NOTE ON DISTANCE INCREASING REDUCIBILITY

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Given two sets of natural numbers $A$ and $B$, $A$ is distance increasing (d. i.) reducible to $B$ if either $A = B$, or if there exists a recursive function $f$ such that $f(n) > n$ for all $n$, and $a \in A$ if and only if $f(a) \in B$. We first show that the analog of the Myhill Isomorphism Theorem holds for distance increasing reducibility. Namely, if $A$ and $B$ are each d. i. reducible to the other, then they are recursively isomorphic. Our main theorem is that there exist nonrecursive r. e. sets $A$ and $B$ which are recursively isomorphic but neither of which is distance increasing reducible to the other.

1. INTRODUCTION

In this paper $N$ represents the set of natural numbers. All functions discussed are defined on a subset of $N$ with range in $N$. A function $f$, defined on $S$, is called partial recursive if there is a Turing machine or program which halts if and only if its input is a member of $S$, and for all $n \in S$, outputs $f(n)$. Partial recursive functions with domain $N$ are called recursive or computable. A set of natural numbers is recursively enumerable or r. e. if it is the range of some recursive function. A set is recursive if its characteristic function is recursive.

Recall that a ‘reducibility’ is a reflexive and transitive binary relation on the subsets of $N$. Two important reducibilities are $m$-reducibility and $1$-reducibility. Given two sets of natural numbers $A$ and $B$, $A$ is said to be $m$-reducible to $B$, written $A \leq_m B$ if there is a recursive function $f$ such that $f(n)$ is in $B$ if and only if $n$ is in $A$. $A$ is $1$-reducible, written $A \leq_1 B$, if the function $f$ may be chosen to be also 1-1. (See Odifreddi², for an extensive survey on several well-studied reducibilities).

Let us define a recursive function $f$ to be distance increasing if $f(n) > n$ for all $n$. Let us further define $A$ to be d.i.-reducible to $B$, written $A \leq_{\text{d.i.}} B$, if either $A = B$ or if $A \leq_m B$ via $f$, for some distance increasing function $f$. It is clear this is a reducibility.

We mention that the notion of distance increasing resembles the notion of length increasing studied in Berman and Hartmanis¹, but in the latter case functions were defined on strings over an alphabet, and were required to increase the length of the string.
The purpose of this note is to examine some interesting relationships between d.i.-reducibility and 1-reducibility. While it is clear that both 1-reducibility and d.i. reducibility are special cases of m-reducibility, it is also easy to see that neither 1-reducibility nor d.i.-reducibility is a consequence of the other. For let \( A = \{ 0, 1 \} \) and \( B = \{ 2 \} \). Then \( A \leq_{d.i.} B \), but \( \not\exists A \leq_1 B \) since \( A \) has larger cardinality than \( B \). Also we have \( B \leq_1 A \), but \( 7 \ B \leq_{d.i.} A \) since a member of \( B \) is greater than all members of \( A \).

Although a distance increasing function is not necessarily 1–1, it comes "close" to being 1–1 in the sense that \( f^{-1} (\langle n \rangle) \) is always a finite set. In the next section we observe another similarity between 1-reducibility and d.i.-reducibility. In the last section, we illustrate a fundamental difference.

2. Myhill's Isomorphism Theorem

An important theorem of Myhill\(^2\) states: If \( A \leq_1 B \) and \( B \leq_1 A \) then there is 1–1 recursive function \( f \) which maps \( N \) onto \( N \) and \( A \) onto \( B \). In this case \( A \) and \( B \) are said to be recursively isomorphic and we write \( A \equiv B \). This theorem is often said to be an analog to the well-known Schroeder-Bernstein Theorem\(^3\) of set theory which states that if \( A \) and \( B \) are (arbitrary) sets such that each can be mapped with 1–1 functions into a subset of the other, then there is a one-to-one correspondence between \( A \) and \( B \). An analog of this theorem also holds for d.i.-reducibility:

**Proposition**—Let \( A \) and \( B \) be sets of natural numbers, and assume \( A \leq_{d.i.} B \) and \( B \leq_{d.i.} A \). Then there is 1–1 recursive function which maps \( N \) onto \( N \) and \( A \) onto \( B \).

**Proof**: The proposition is obvious in the case when \( A = B \), so assume \( A \neq B \). Then there are distance increasing recursive functions \( f \) and \( g \) such that \( A \leq_{d.i.} B \) via \( f \) and \( B \leq_{d.i.} A \) via \( g \). By Myhill's theorem, it suffices to show \( A \leq_1 B \) and \( B \leq_1 A \). To show \( A \leq_1 B \) we construct a 1–1 recursive function \( h \) such that \( x \in A \) if and only if \( h (x) \in B \). The recursive function \( h \) is defined inductively. First let \( h (0) = f (0) \), and now assume that each \( h (i) \) has been defined for \( i < k \). For \( i = 0, 1, 2 \ldots \) define \( \{ n_i \} \) to be the sequence

\[
f(k), f \left( g \left( f \left( k \right) \right) \right), \ldots \]

We note that all members of the sequence are distinct since the sequence is strictly increasing. Define \( h (k) \) to be the first term in the sequence not equal to any of the previously defined values \( h (i) \), for \( i < k \). It is clear that \( h \) is both recursive and 1–1. It remains to show that \( x \in A \) if and only if \( h (x) \in B \). However by our assumption of \( f \) and \( g \) we know

\[
x \in A \iff f (x) \in B \iff g \left( f (x) \right) \in A \iff f \left( g \left( f (x) \right) \right) \in B \ldots
\]

and so \( A \leq_1 B \). By a similar argument we can show \( B \leq_1 A \).

3. Recursively Enumerable Sets

In this section we construct nonrecursive r.e. sets \( A \) and \( B \) such that \( A \equiv B \) but neither \( A \) nor \( B \) is d.i.-reducible to the other. We assume \( \{ \phi_e \}, e = 0, 1, 2, \ldots \) is a
standard numbering of the partial recursive functions. We write \( \phi_{es} (x) \downarrow \) to mean that the \( e \)th Turing machine, which defines \( \phi_e \), halts for input \( x \) in less than \( s \) steps to some output value \( y \), where \( e, x \) and \( y \) are each less than \( s \).

**Lemma 1**—If \( A \) is a recursive set and both \( A \) and \( \sim A \) are infinite, then there is a distance increasing function \( f \) such that \( A \leq_{di} A \) via \( f \).

**Proof**: By the assumptions about \( A \) there are recursive functions \( g_1 \) and \( g_2 \) such that

\[
g_1 (0), g_1 (1), g_1 (2), \ldots
\]

and

\[
g (0), g_2 (1), g_2 (2), \ldots
\]

are, respectively, strictly, increasing enumerations of the members of \( A \) and \( \sim A \). We may take \( f (n) \) to be \( g_1 (n + 1) \) if \( n \in A \), and \( g_2 (n + 1) \) otherwise. This is a distance increasing function since \( f (n) = g_1 (n + 1) > n \).

**Theorem**—There exists a nonrecursive r.e. set \( A \) such that for any distance increasing recursive function \( f \), \( A \) is not reducible to itself via \( f \).

**Construction**—We construct \( A \) in stages. At each stage we enumerate at most one new element into \( A \). Hence there will be a recursive sequence which enumerates the set \( A \), and \( A \) will be r.e. The construction makes use of the well-known priority method$^{2,4}$. For each partial recursive function \( \phi_e, e = 0, 1, 2, \ldots \) we must satisfy the requirement

\[
R_e : \forall A \leq_{di} A \text{ via } \phi_e.
\]

That is, we must prevent \( A \) from being \( d.i.- \)-reducible to itself via \( \phi_e \). At the same time we will build \( A \) so that both it and its complement are infinite. Lemma 1 will then insure that \( A \) is nonrecursive.

During the construction, at each new stage \( s \), a restraint function \( r (e, s) \) is updated for each value \( e \). At stage 0, \( r (e, 0) \) is initialized to \( -1 \) for all \( e \). Once \( r (e, s) \) becomes nonnegative, requirement \( R_e \) will be maintained as long as new elements are always greater than \( r (e, s) \). At each stage \( s + 1 \) we try not to injure requirements \( R_e \) and hence try to keep incoming elements greater than \( r (e, s) \). This is not always possible, and like typical "priority arguments" is given to lower numbered requirements.

During the construction which follows, odd stages will be used to insure \( A \) and \( \sim A \) are infinite, and even stages will be used to meet the requirements \( R_e \). Elements for \( A \) are always drawn from two infinite recursive sets \( W \) and \( W' \). We assume

\[
W' = \{n_0 < n_1 < n_2 < \ldots\}
\]

and that \( W \) and \( W' \) are disjoint. The set \( A_i \) designates the elements enumerated into \( A \) at the end of stage \( i \).
Stage 0: Initialize $A_0 = \{\}$ and $r(e, 0) = -1$ for all $e$.

Stage $2s + 1$: Enumerate into $A$ the least $n_i \in W' - A_{2s}$ such that $i$ is odd and $n_i > r(e, 2s)$ for all $e$. Set $r(e, 2s + 1) = r(e, 2s)$ for all $e$.

Stage $2s + 2$: Agree that $R_e$ requires attention if

(a) $r(e, 2s + 1) = -1$, and

(b) There is some $x_0 \in W - A_{2s+1}$ such that $x_0 < \phi_{e,s}(x_0)$ and $x_0 > r(e', 2s+1)$ for all $e' < e$.

If no requirement at this stage requires attention then define $A_{2s+2} = A_{2s+1}$ and $r(e', 2s + 2) = r(e', 2s + 1)$ for all $e'$. Otherwise let $e$ be smallest such that $R_e$ requires attention, and let $x_0$ be a number satisfying (b). Then $R_e$ receives attention and we do the following:

Case 1—$\phi_e(x_0) \in A_{2s+1}$. Define $r(e, 2s + 2) = x_0 + 1$, and for all other $e'$, let $r(e', 2s + 2) = r(e', 2s + 1)$. Hence we try to keep $x_0$ from entering $A$.

Case 2—otherwise. Enumerate $x_0$ into $A$. Define $r(e, 2s + 2) = \phi_{e,s}(x_0) + 1$. Hence we try to keep $\phi_e(x_0)$ from entering $A$. For $e' < e$ keep $r(e', 2s + 2)$ the same as $r(e', 2s + 1)$ but for $e' > e$ reset $r(e', 2s + 2) = -1$.

The following lemmas complete the proof of the theorem.

Lemma 2—Each requirement $R_e$ receives attention at most $2^e$ times.

Proof: Note $r(e, s) = -1$ and $r(e, s + 1) \geq 0$ if and only if $R_e$ receives attention at stage $s + 1$. Furthermore, $r(e, s) \geq 0$ and $r(e, s + 1) = -1$ if and only if $R_{e_i}$, for some $e_i < e$, received attention at stage $s + 1$. It follows that $R_0$ can receive attention at most once. Assume by induction that the lemma is true for all $e_1 < e$, and let $s_1, s_2, \ldots$ be the stages at which $R_{e_i}$ receives attention. Then for each $i > 1$ there is some $e_i < e$ for which $R_{e_i}$ receives attention between stages $s_{i-1}$ and $s_i$. By our induction assumption, after $R_e$ receives attention for the first time, it can only receive attention $2^e - 1$ more times. This completes the proof.

Lemma 3—Assume $r(e, s) \geq 0$ and for each $e' < e$, $R_{e'}$ never receives attention after stage $s$. Then $A$ satisfies requirement $R_e$.

Proof: Let $s_0$ be the last stage at which $R_e$ receives attention. By our comments above, $s_0 \leq s$, and $s_0$ is the last stage at which any $R_{e'}$ $e' \leq e$, receives attention. At the end of stage $s_0$, there is a number $x_0 < \phi_e(x_0)$ such that exactly one member of the pair $(x_0, \phi_e(x_0))$ has been placed in $A$. Requirement $R_e$ will be satisfied if the other member of the pair does not ever enter $A$. However, at the end of stage $s_0$, $r(e, s_0)$ is defined to be greater than the member in the pair not enumerated into $A$. Since at every stage following $s_0$, any new element enumerated into $A$ is always greater than $r(e, s_0)$, the lemma follows.
Lemma 4—A and \( \sim A \) are infinite and all requirements \( R_e \) are satisfied.

Proof: \( A \) is infinite since at each odd numbered stage we enumerate an element from \( W' \sim A \) is infinite since we exclude from \( A \) all \( n_i \) for even \( i \). Let \( e \) be fixed, and we now show \( R_e \) is satisfied. We may assume for all \( x \phi_e(x) \) is defined and \( \phi_e(x) > x \). By Lemma 2 we may choose \( s \) so that no requirement \( R_{es} \), for \( e < e' \), ever receives attention at stage \( s \) or later. If \( r(e, s) \geq 0 \) then \( R_e \) will be satisfied by Lemma 3. Assume, then, \( r(e, s) = -1 \). Choose \( x_0 \in W - A_s \) such that \( x_0 \geq r(e', s) \) for all \( e' < e \). Now let \( s' \geq s \) be minimal such that \( \phi_{es'}(x_0) \downarrow \). Then \( R_e \) requires and receives attention no later than stage \( 2s' + 2 \), and so \( r(e, 2s' + 2) \) becomes non-negative. Again, by Lemma 3, \( R_e \) will be satisfied.

We now may prove the result mentioned in the beginning of this section.

Corollary—There exist nonrecursive r.e. sets \( A \) and \( B \) such that \( A \equiv B \) but neither \( A \leq_{dl} B \), nor \( B \leq_{dl} A \).

Proof: Let \( A \) be the set constructed in the previous theorem. Let \( a_0 \) be the smallest number in \( A \), and \( a_0' \) the smallest number not in \( A \). Let \( B = A \cup \{a_0\} \setminus \{a_0'\} \). Then \( A \) and \( B \) are easily seen to be recursively isomorphic via the function which maps \( a_0' \) to \( a_0 \), \( a_0 \) to \( a_0' \), and leaves all other numbers fixed. However suppose \( A \leq_{dl} B \) via the distance increasing function \( f(x) \). Then let \( C \) be the finite set of \( x \) for which \( f(x) = a_0 \), and \( C' \) the finite set of \( x \) for which \( f(x) = a_0' \). Let \( a_1 \) be the smallest element in \( A \), greater than \( a_0' \), and let \( a_1' \) be the smallest element not in \( A \), but greater than \( a_0 \). Then let \( f_1 \) be the function which maps \( x \) to \( a_1' \) for \( x \in C \), \( x \) to \( a_1 \) for \( x \in C' \), and all other \( x \) to \( f(x) \). It is easy to see that \( f_1 \) is distance increasing and that \( A \) is d.i.-reducible to itself via \( f_1 \), a contradiction. A similar argument may be used to show \( \sim B \leq_{dl} A \).

References