ON RELATIVE TOPOLOGICAL DEGREE OF SET-VALUED
COMPACT VECTOR FIELDS

LJ. GAJIC

Prirodno-matematički fakultet, Institut za matematiku, 21000 Novi Sad, Dr Ilije
Djuricica 4, Jugoslavija

(Received 30 December 1986)

The object of this paper is to define a concept of relative topological degree for
a class of set-valued compact vector fields with respect to a closed convex
subset in a topological vector space. Some usuały properties of this concept
have been investigated too.

INTRODUCTION

The concept of relative topological degree of single-valued compact vector fields
with respect to a closed convex subset in a locally convex topological vector space was
introduced by Borisovich¹. Duc et al.² generalized this concept to the set-valued case.
In this paper, we propose to generalize this concept for a class of set-valued compact
vector fields in not necessarily locally convex linear topological spaces.

Using the concept of relatively topological degree we define a concept of topolog-
ical degree for related class of ultimately compact set-valued vector field without
retraction property of Petryshyn and Fitzpatrick³.

The paper consists of four sections. The first section sets the notations and con-
tains basic results for finite dimensional reductions. The second section is devoted to a
theory of relative topological degree in linear topological space of a class of set-valued
compact vector fields. In Section 3, we shall consider the topological degree for related
class of ultimately compact set-valued vector fields. The fourth and last section contains
two fixed point theorems.

1. PRELIMINARY RESULTS

Let $X$ be a Hausdorff topological vector space (HTVS) over the real numbers
fields $\mathbb{R}$; $\mathcal{H}(X)$ the family of non-void closed convex subsets of $X$; $\mathcal{H}(0)$ the family
of balanced symmetric neighbourhoods of zero; $L$ a finite-dimensional vector subspace
of $X$, $\mathcal{L}(0)$ the family of closed symmetric convex neighbourhoods of 0 in $L$; $D$ a
nonvoid open subset of $X$. 
Following are some basic definitions and properties of compact vector fields in topological linear spaces. Let $T$ be a map from topological space $Y$ into $\mathcal{H} (X)$, let $A$ be a subset of $Y$, and define $T (A) = \bigcup_{x \in A} T (x)$. $T$ is said to be upper semi continuous (u.s.c. for short) on $Y$ iff for each $B \subset Y$ and each open $W \subset X$ with $T (B) \subset W$, there exists an open set $V$ in $Y$ such that $B \subset V$ and $T (V) \subset W$. A map $T$ of $Y$ into $\mathcal{H} (X)$ is said to be compact iff $T$ is u.s.c. on $Y$ and $T (Y)$ is relatively compact (i.e. $T (Y)$ is compact). A compact vector field on $\mathcal{D}$ (to $\mathcal{H} (X)$) is a map of $\mathcal{D}$ into $\mathcal{H} (X)$ of the form $I - T ((I - T) (x) = x - T (x))$ where $T$ is a compact map of $Y$ into $\mathcal{H} (X)$.

**Definition** 15—A subset $C \subset X$ is of $Z$-type iff for every $U \in \mathcal{H} (0)$ there exists a $V = V (U) \in \mathcal{H} (0)$ such that:

$$\text{conv} (V \cap (C - C)) \subset U.$$  \hfill ...(1)

(conv = convex hull).

**Remark 1:** Every subset in locally convex space is of $Z$-type. For another, non-trivial example, see Hadzic\(^2\).

**Remark 2:** It is easy to prove that for every $U \in \mathcal{H} (0)$ there exists $V = V (U) \in \mathcal{H} (0)$ such that:

$$\text{conv} ((A + V) \cap C) \subset A + U$$ \hfill ...(2)

for every convex subset $A$ of $Z$-type subset $C$ of $X$.

Hadzic\(^3\) proved the following generalization of Leray-Schauder-Nagumo lemma for $Z$-type subsets.

**Lemma** 15—Let $C$ be a compact $Z$-type subset of $X$. Then for every $U \in \mathcal{H} (0)$ there exists a continuous map $\rho$ of $C$ into $X$ and a finite set $B \subset C$ such that:

$$\rho (C) \subset \text{conv} B$$

and

$$\rho (x) - x \in U \text{ for every } x \in C.$$

**Remark:** The map $\rho$ in this lemma is in fact, a so called Schauder projection\(^3\).

**Proposition 1**—Let $Y$ be a Hausdorff space, $X$ a HTVS and $T : Y \to 2^X$ compact u.s.c. mapping with $T (x)$ a closed convex non-empty subset, $T (Y) \subset C_0$, $C_0$ convex $Z$-type subset. For $U \in \mathcal{H} (0)$ let $\rho : \overline{T (Y)} \to I$ be a Schauder projection into a finite-dimensional linear subspace $L$ such that:

$$\rho (y) - y \in V = V (U)$$

for each $y \in \overline{T (Y)}$ and for each $x \in Y$ let:

$$P \rho T (x) = \text{conv} \rho (T (x)).$$
Then:

(a) \( \text{conv } PT (x) \) and each \( PT (x) \) is compact and convex subset of \( C_0; \)

(b) \( PT : Y \to 2^X \) is u.s.c and finite-dimensional;

(c) \( PT (x) \subseteq T(x) + U \) for each \( x \in Y. \)

**Proof**: (a) As in Dugundji³, because \( p (T(x)) \subseteq L (\dim L < + \infty) \) is compact it's convex closure \( PT (x) \) is a compact and convex subset of \( L \) and \( C_0. \) For the same reason, the compactness of \( T(Y) \) implies that of \( \text{conv } PT (Y). \)

(b) Clearly, only that \( PT \) is u.s.c requires proof. Choose any \( x \in Y \) and let \( W \subseteq X \) be open with \( PT (x) \subseteq W. \) Since \( PT (x) \) is compact there is a \( U_1 \in \mathcal{U} (0) \) such that \( PT (x) + U_1 \subseteq W \) and let \( U \in \mathcal{U} (0) \) be so that \( U + U \subseteq U_1. \) Since \( C_0 \) is of Z-type there is a \( V \in \mathcal{U} (0) \) so that

\[
\text{conv } ((C + V) \cap C_0) \subseteq C + U
\]

for every convex subset \( C \) of \( C_0. \) Clearly \( p (T(x)) \subseteq PT (x) \subseteq PT (x) + V. \) Being the composition of two point-compact and u.s.c set functions, \( x \mapsto p T (x) \) is also point-compact and u.s.c, so there is a neighbourhood \( V(x) \in \mathcal{U} (x) \) with \( p T (x) \subseteq PT (x) + V \) for all \( y \in V(x). \) According to the choose of \( V \) we find:

\[
PT (y) = \text{conv } (p T (y) \cap C_0) \subseteq \text{conv } ((PT (x) + V) \cap C_0)
\]

\[
\subseteq PT (x) + U \subseteq PT (x) + U_1 \subseteq W
\]

for all \( y \in V(x), \) so \( PT (V) \subseteq W \) and because \( x \) is arbitrary \( PT \) is u.s.c.

(c) Let \( z \in PT (x). \) We have \( z = \sum_{i=1}^n \lambda_i z_i \) for suitable \( z_i \in p T(x) \) and real

\( 0 \leq \lambda_i \leq 1 \) with \( \sum_{i=1}^n \lambda_i = 1. \) For each \( i \) choose \( y_i \in T(x) \) so that \( p(y_i) = z_i. \) Then

\[
p(y_i) - y_i = v_i \in V (i = 1, 2, \ldots, n)
\]

so we have \( y_i + v_i \in T(x) + V \) for each \( i \) and

\[
z = \sum_{i=1}^n \lambda_i p(y_i) = \sum_{i=1}^n \lambda_i (y_i + v_i)
\]

\( \in \text{conv } ((T(x) + V) \cap C_0) \subseteq T(x) + U. \)

2. A Degree Theory

Throughout this section, \( T \) is u.s.c map of \( D \) into \( \mathcal{K}(X) \) and \( K \) is a closed convex subset of \( X \) such that:

(i) \( H = \text{cl } T(K \cap D) \) is compact,
(ii) \( H \subseteq K \)

(iii) \( 0 \in X \setminus (I - T) (K \cap \partial D) \),

(iv) \( H \) is contained in some convex Z-type subset \( C_0 \) of \( X \).

As in Ma's paper\(^8\) one can prove:

**Proposition 2**—Let \( Y \) be a closed subset of \( D \). Then \( (I - T)(K \cap Y) \) is closed. In particular, there exists a \( V \in \mathcal{C}(0) \) such that :

\[
V \cap (I - T)(K \cap \partial D) = \emptyset.
\]

...(3)

Let \( V_1 \in \mathcal{C}(0) \) be such that :

\[
V_1 \subset V
\]

...(3')

where \( V \) is from Proposition 2.

Assume that \( D \cap K \neq \emptyset \). With \( PT \) as in Proposition 1 for \( U = V_1 \) (as in (3')) we have that :

\[
0 \notin (I - PT)(K \cap \partial D).
\]

Now, let \( L \) be a finite dimensional vector subspace of \( X \) such that \( p \circ (H) \subseteq L \) and \( L \cap K \cap D \neq \emptyset \). Then \( L \cap K \) is closed convex subset of finite dimensional vector space \( L \). Hence, by Tietze’s theorem there exists a continuous map \( f \) of \( L \) into \( K \cap L \) such that \( f(x) = x \) for each \( x \in L \cap K \). Let \( U \) be a relativity open subset of \( L \) such that :

\[
L \cap K \cap D \subseteq U \subseteq f^{-1}(L \cap K \cap D).
\]

As in Duc et al.\(^9\) one can prove that \( 0 \notin (I - PTf)(\partial U) \) and that \( PT f \; |_U \) is a compact map on \( U \) to \( \mathcal{H}(L) \) so \( \deg (U, 0, I - PTf) \) is defined. Whence we have the following definition:

**Definition 2**—Let \( T \) be u.s.c map of \( \overline{D} \) into \( \mathcal{H}(X) \). Suppose \( K \) is a closed convex subset of \( X \) such that (i), (ii), (iii), (iv) are satisfied.

We pose :

\[
D_K (D, 0, I - T) = \begin{cases} 
0 & \text{if } K \cap D = \emptyset, \\
\deg (U, 0, I - PTf) & \text{if } K \cap D \neq \emptyset,
\end{cases}
\]

where \( \deg (U, 0, I - PTf) \) is topological degree in finite dimensional vector space.

We shall say that \( D_K (D, 0, I - T) \) is the degree at \( 0 \) of \( I - T \) on \( D \) relative to \( K \).

Similarly as in Duc et al.\(^9\) we can prove that \( D_K (D, 0, I - T) \) does not depend on the choice of \( PT, L, f \) and \( U \) so this concept is well defined.
**Remark**: For $K = E$ this definition is given in Gajic since (iv) imply that $T$ is uniformly finite approachable map.

Now, we shall show that $D_K(D, 0, I - T)$ enjoys usually properties. At first

**Theorem 1**—Suppose $D_K(D, 0, I - T) \neq 0$. Then there exists an $x$ in $D \cap K$ such that $x \in T(x)$.

**Proof**: As in Duc et al.².

To proceed with study of the properties of $D_K$, we shall need the following extension of the Leray-Schauder-Nagumo lemma.

**Lemma 2**—Let $H_1 \subset H_3$ be two nonvoid compact subsets of convex $Z$-type subset $H \subset X$ and let $U \in \mathcal{Y}(0)$. Then there exist two open sets $W_1 \subset W_2$ in $X$ such that $H_1 \subset W_1$, $H_2 \subset W_2$ and a continuous map $p$ of $W_2$ into $\text{conv} B$, $B$ finite subset of $H_2$, such that

1. $x - p(x) \in U$ for each $x \in H \cap W_2$,
2. $p(W_i) \subset \text{conv} H_1$.

**Proof**: Let be $V = V(U) \in \mathcal{Y}(0)$ such that (1) (for $C = H$) is valid and $V_1 \in \mathcal{Y}(0)$, $V_1 \subset V$. Further, let $\{a_1, a_2, \ldots, a_m\} \subset H_1$ and $\{a_m, \ldots, a_n\} \subset H_2 \setminus \sum_{j=1} V_1$ be such that

$$H_1 \subset W_1 := \bigcup_{j=1}^m (a_j + V_1)$$

and

$$H_2 \setminus \bigcup_{j=1}^m (a_j + V_1) \subset \bigcup_{j=m+1}^n (a_j + V_1).$$

Let

$$W_2 := (\bigcup_{j=1}^m (a_j + V_1)) \cup (\bigcup_{j=m+1}^n (a_j + V_1)).$$

It seen that $W_1, W_2$ are open, $W_1 \subset W_2$, $H_1 \subset W_1$, $H_2 \subset W_2$. Let $\{q_j\}_{j=1}^n$ be a partition of unity for cover in (4) and

$$p(x) = \sum_{j=1}^n q_j(x) a_j, x \in W_2.$$  

For $x \in W_2 \cap H$ we have:

$$x - p(x) = \sum_{i=1}^n q_i(x) (x - a_j)$$

(equation continued on p. 274)
As in (Duc et al.\textsuperscript{2} one can prove 2).

**Definition 3**—Let $T$, $K$ be as in Definition 2. Put $K (T, D, K, 0) = K$

\[
K (T, D, K, j) = \underbrace{\text{conv} \ T (\bar{D} \cap K (T, D, K, j - 1))}_{i < j} \text{if } (j - 1) \text{ exists, } K (T, D, K, j)
\]

\[
\cap K (T, D, K, i) \text{ if } (j - 1) \text{ does not exists.}
\]

If $T$, $K$ are as in Definition 2 it is not difficult to prove Lemma 4, Proposition 5, Proposition 6. Theorem 3 as in Duc et al.\textsuperscript{2}.

Similarly as in Duc et al.\textsuperscript{2} one can prove:

**Theorem 2**—Let $F$ be a u.s.c map of $[0, 1] \times \bar{D}$ into $\mathcal{K} (X)$ with following properties:

1. $0 \notin x - F (t, x)$ for every $(t, x) \in [0, 1] \times (K \cap \partial D)$;
2. $H = \text{cl} F (J \times (K \cap \bar{D}))$ is a compact subset of $K$, $J = [0, 1]$;
3. $H$ is contained in some convex Z-type subset $C_0$ of $X$.

Put $F_t (x) = F (t, x)$ for all $(t, x) \in [0, 1] \times \bar{D}$.

Then $D_K (D, 0, I - F_t)$ is defined for each $t \in [0, 1]$ and

\[
D_K (D, 0, I - F_t) = D_K (D, 0, I - F_0).
\]

3. **Topological Degree of Ultimately Compact Vector Fields**

The concept of ultimately compact vector fields was introduced by Sadovski\textsuperscript{9}. This concept has been generalized by Petryshyn and Fitzpatrick\textsuperscript{7} to set-valued vector fields in locally convex linear topological spaces in which closed convex sets are retracts. We shall rely on results in Section 2 on $K$-degree to define the concept of degree for one class of ultimately compact set-valued vector fields in general linear vector spaces without the retraction condition.

**Definition 4**—Let $Y$ be a topological space and let $F$ be an u.s.c map of $Y \times \bar{D}$ into $\mathcal{K} (X)$. Define, for every ordinal $i$

\[
K (F, Y \times \bar{D}, 0) = \text{conv} \ F (Y \times \bar{D}).
\]

\[
K (F, Y \times \bar{D}, i) = \text{conv} \ F (Y \times (\bar{D} \cap K (F, Y \times D, i - 1)))
\]

if $(i - 1)$ exists.
Set-Valued Compact Vector Fields

\[ K(F, Y \times \overline{D}, i) = \bigcap_{j < i} K(F, Y \times \overline{D}, j) \text{ if } (i - 1) \text{ does not exists.} \]

If no confusion can arise, we shall write \( K_i \) for \( K(F, Y \times \overline{D}, i) \). If \( u \) is an ordinal strictly larger than the cardinal of \( \text{conv} \ F(Y \times \overline{D}) \), then as is easily seen \( K_i = K_u \) for every ordinal \( i > u \).

If \( u \) is such an ordinal we put
\[ K(F, Y \times \overline{D}, u) = K_u. \]

We shall say \( F \) is \( Y \)-ultimately compact map of \( Y \times \overline{D} \) into \( \mathcal{H}(X) \) if \( \text{cl} \ F(Y \times (\overline{D} \times K(F, Y \times \overline{D}))) \) is compact.

If \( Y \) is a singleton, we identify \( F \) with a map of \( \overline{D} \) into \( \mathcal{H}(X) \), in this case if \( F \) is \( Y \)-ultimately compact, we shall say \( F \) is ultimately compact, for short.

**Definition 5**—Let \( T \) be an ultimately compact map of \( \overline{D} \) into \( \mathcal{H}(X) \) such that \( 0 \not\in X \) \( (I - T) \) \( (\partial \overline{D}) \) and \( T(\overline{D}) \) is contained in some convex subset of \( Z \)-type. For \( K = K(T, \overline{D}) \), we see from Definition 4 that \( K \) has all the properties of the \( K \) in Definition 2. Hence that \( D_{K(T, \overline{D})}(D, 0, I - T) \) is defined. We say \( D_{K(T, \overline{D})}(D, 0, I - T) \) is the topological degree of the ultimately compact vector field \( I - T \).

Whit \( T, K \) as in above (Definition 5), Theorems 5 and 6 of Duc et al.\(^2\), Theorem 6\(^3\) are valid and the next theorem is valid too.

**Theorem 3**—Let \( F \) be a \( Y \)-ultimately compact map of \( Y \times \overline{D} \) into \( \mathcal{H}(X) \) such that \( 0 \notin X - F(t, x) \) for \( (t, x) \in Y \times \overline{D} \). If we suppose that \( F(Y \times \overline{D}) \) is contained in some convex \( Z \)-type subset then:

1. \( F \) is ultimately compact map into \( \mathcal{H}(X) \),
2. \( D_{K(F_0, \overline{D})}(N, 0, I - F_0) = D_{K(F_1, \overline{D})}(D, 0, I - F_1). \)

4. Some Fixed Point Theorems

We shall conclude this paper with fixed point theorems.

**Theorem 4**—Let \( A \) be a nonvoid convex not necessarily closed \( Z \)-type subset of \( X \) and let \( T \) be a compact map of \( A \) into \( \mathcal{H}(X) \) such that \( \text{cl} \ T(A) \subseteq A \). Then \( T \) has a fixed point in \( A \).

**Proof**: Similarly as in Duc et al.\(^2\) but using Proposition 1.

**Theorem 5**—Let \( B \) be a nonvoid convex (not necessarily closed) \( Z \)-type subset of \( X \), let \( T \) be an ultimately compact map of \( B \) into \( \mathcal{H}(X) \) such that \( \text{cl} \ T(B) \subseteq B \) and \( K(T, B) \neq \phi \). Then \( T \) has a fixed point.

**Proof**: As in Duc et al.\(^2\) but using Theorem 4.
REFERENCES

3. J. Dugundji, and A. Granas, Fixed Point Theory, Volume 1, Warszawa, 82.