ON FARTHEST POINT PROBLEM

B. B. PANDA

Department of Mathematics, Sambalpur University, Jyoti Vihar
Burla, Sambalpur, Orissa 768017

(Received 21 January 1987; after revision 30 April 1987)

In this paper, it is shown that every uniquely remotal set in spaces like \( L^1 \), \( l^1 \) and \( L^\infty [0, 1] \) is a singleton. The same result is also shown to be true in subalgebras of \( C(\Omega) \), where \( \Omega \) is compact, and in certain infinite dimensional subspaces of a Hilbert space with a modified norm. The problem has been also explored in a subspace \( Y \) of \( C[0, 1] \) which contains congruent images of all separable Banach spaces.

1. Introduction

The present paper considers the following farthest point problem:

'Let \( K \) be a uniquely remotal set in a normed linear space \( X \). Then is it necessarily a singleton?'

Recall that a nonempty bounded set \( K \) in a normed linear space \( X \) is called remotal (respectively uniquely remotal) if the map \( q : X \to 2^K \) defined by \( q(x) = \{ y \in K : \| x - y \| = \sup_{z \in K} \| x - z \| \} \) is nonempty (respectively singleton) for every \( x \) in \( X \). An element belonging to \( q(x) \) is called a farthest point from \( x \). The function \( F_K(x) = \sup_{y \in K} \| x - y \| \) is called the farthest distance function associated with \( K \).

Some partial and affirmative answers to the problem are known\(^1\)\(^-\)\(^9\). In the case of a Hilbert space, the problem reduces to answering another famous but unsolved problem, namely, the convexity problem of Chebyshev sets (see Klee\(^7\)). Some general infinite dimensional spaces admitting an affirmative answer to the problem are \( c_0 \), \( c \) and \( C(\Omega) \), where \( \Omega \) is compact and Hausdorff. A few more cases are given by Bosznay\(^8\) (preprint). In the present paper the problem has been solved in spaces like \( L^1 [0, 1] \), \( l^1 \) and \( L^\infty [0, 1] \). It has been also shown that the problem admits an affirmative answer in subalgebras of \( C(\Omega) \) and in certain subspaces of a Hilbert spaces with a modified norm thus improving some of the known results of Bosznay\(^8\) proved under more restricted conditions. The problem has been also explored in a certain subspace \( Y \) of \( C[0, 1] \) which contains the congruent images of all separable Banach spaces.

2. Main Results

To begin with, we consider the farthest point problem in the space \( L^1 [0, 1] \). Accordingly, we denote
\[ A_{n,l} = \left( \frac{i}{n}, \frac{i+1}{n} \right), i = 0, 1, ..., n - 1 \]

\[ Y_n = \text{span} \{ \chi_{A_{n,l}} : 0 \leq i \leq m - 1, 1 \leq m \leq n \} \]

and

\[ Y = \text{span} \{ \chi_{A_{m,l}} : 0 \leq i \leq m - 1, \text{and for all } m \} \]

where \( \chi_A \) denotes the characteristic function of the set \( A \).

**Theorem 1**—Every uniquely remotal set in \( L^1 [0, 1] \) is a singleton.

**Proof** : Let \( K \) be a uniquely remotal set in \( L^1 [0, 1] \). Let \( x \in Y \); then \( x \in Y_n \) for some \( n \) and it can be written in the form \( x = \alpha_1 \chi_{B_1} + ... + \alpha_m \chi_{B_m} \), where \( B_m \)'s \( 1 \leq m \leq n \left( \frac{n+1}{2} \right) \) are pairwise disjoint sets obtained in the usual way from \( A_{n,l} \)'s satisfying \( \bigcup_{m=1}^{n(n+1)/2} B_m = \bigcup \{ A_{n,l} : 0 \leq i \leq m - 1, 1 \leq m \leq n \} \). Let \( A \) and \( B \) be any two disjoint sets obtained by taking union of members of the collection \( B_m \). When \( n \) is fixed, there are only a finitely many pairs \( (A, B) \) and therefore the total number of such pairs \( (A, B) \) is countable when \( n \) ranges over \( N \). We shall now say that an \( x \in Y \) is in \( K (A, B) \) if

\[
\int_A (x - q(x)) \, d\mu \geq 0,
\]

\[
\int_B (x - q(x)) \, d\mu \leq 0
\]

and \( x(t) = 0 \) ae. on \( (A \cup B)^c \). Note that one of \( A \) and \( B \) could be also empty. It will be first shown that \( q(x_1) = q(x_2) \) whenever \( x_1, x_2 \in K (A, B) \). Accordingly, define \( x_3 \in Y \) as follows:

\[
x_3(t) = \begin{cases} 
\max \{ x_1(t), x_2(t) \} & \text{if } t \in A \\
\min x_1(t), x_2(t) & \text{if } t \in B \\
0 & \text{if } t \notin A \cup B.
\end{cases}
\]

Then, by definition,

\[
\| x_3 - q(x_1) \| = \int_A (x_3 - q(x_1)) \, d\mu + \int_B (q(x_1) - x_3) \, d\mu
\]

\[
+ \int_{[0,1]} |q(x_1)| \, d\mu
\]

\[
= \int_A (x_3 - x_1) \, d\mu + \int_A (x_1 - q(x_1)) \, d\mu + \int_B (q(x_1) - x_3) \, d\mu
\]

\[
+ \int_B (x_1 - x_3) \, d\mu + \int_{[0,1]} |q(x_1)| \, d\mu
\]

\[
= \| x_3 - x_1 \| + \| x_1 - q(x_1) \| \text{ for } i = 1, 2.
\]
But \( F_K (x_3) - F_K (x_i) \leq \|x_3 - x_i\| \) and this, coupled with the unique remotal property of \( K \), implies that \( q (x_i) = q (x_3) \) for \( i = 1, 2 \) and this is what it was claimed earlier.

Now, observing that the number of \( K (A, B) \) so constructed is countable and that any \( x \in Y \) belongs to at least one \( K (A, B) \), we conclude that the subspace \( Y \) admits only countably many farthest points in \( K \). But, as in Lemma 3 of Asplund\(^1\) any one dimensional subspace \( L \) of \( Y \) is then union of a countable collection \( \{L \cap q^{-1} (x)\} \) (empty sets being discarded) of disjoint closed sets in \( L \) which is an impossibility since the real line (a homeomorphic of \( L \)) cannot be covered by a countable collection of disjoint closed sets. Thus \( Y \) must admit a single farthest point, say, \( q (x_0) \) in \( K \).

Finally, we observe that \( Y \) is dense in \( L^1 [0, 1] \) and hence for any \( x \in L^1 [0, 1] \), there is a sequence \( x_n \in Y \) such that \( x_n \to x \). But \( F_K (x_n) = \|x_n - q (x_0)\| \), and by the continuity of \( F_K (x) \), we obtain \( F_K (x) = \|x - q (x_0)\| = \|x - q (x_0)\| \). As \( K \) is uniquely remotal, this leads to \( q (x) = q (x_0) \) for all \( x \in L^1 [0, 1] \). This implies that \( K \) must be a singleton.

In particular, when the Lebesgue measure is replaced by the counting measure, the sets \( A, B \) can be taken to be finite but disjoint subsets of \( N \) and the subspace \( Y \) is then taken to be the span of \( \{e_1, e_2, \ldots, e_n, \ldots\} \). We now set \( x \in K (A, B) \) if

\[
(x - q (x)) (n) \geq 0 \quad \text{for} \quad n \in A \\
< 0 \quad \text{for} \quad n \in B
\]

and

\[
x (n) = 0 \quad \text{for} \quad n \notin A \cup B.
\]

The following result, a partial answer to which has been given by Theorem 2 of Bosznay\(^2\), now follows immediately.

**Theorem 2**—Every uniquely remotal set in \( L^1 \) is a singleton.

**Theorem 3**—In \( L^\infty [0, 1] \), every uniquely remotal set is a singleton.

**Proof**: Let \( K \) be a uniquely remotal set in \( L^\infty [0, 1] \). For \( x, y \in L^\infty [0, 1] \), define

\[
A_n = \{ t \in [0, 1] : \| x (t) - q (x) (t) \| \geq F_K (x) - \frac{1}{n} \},
\]

and

\[
B_n = \{ t \in [0, 1] : \| y (t) - q (y) (t) \| \geq F_K (y) - \frac{1}{n} \}.
\]

There is no loss of generality in assuming that \( F_K (x) = F_K (y) \). For, if \( F_K (x) > F_K (y) \), then \( \lambda > 1 \) can be chosen such that

\[
\| \lambda y + (1 - \lambda) q (y) - q (y) \| = F_K (\lambda y + (1 - \lambda) q (y)) = \lambda F_K (y) = F_K (x).
\]
We have two cases to consider.

**Case I**—Suppose that \( \mu (A_n \cap B_n) = 0 \) for some \( n \). Then define a \( z \in L^\infty [0, 1] \) by setting

\[
  z(t) = \begin{cases} 
  x(t) & \text{if } t \notin B_n \\
  y(t) & \text{if } t \in B_n.
  \end{cases}
\]

Clearly, \( z \) has both \( q(x) \) and \( q(y) \) as farthest points and, by the unique remotal property of \( K \), \( q(x) = q(y) \).

**Case II**—Suppose that \( \mu (A_n \cap B_n) > 0 \) for all \( n \). Denote \( C_n = A_n \cap B_n \). Then the ess. sup of both \( x - q(x) \) and \( y - q(y) \) need only be taken on \( C_n \) to obtain \( F_K(x) \) and \( F_K(y) \) respectively. If \( \mu (C_n) < 1 \), define an \( L^\infty \)-function \( u \) by putting

\[
  u(t) = \begin{cases} 
  \alpha F_K(\theta) & \text{if } t \notin C_n \\
  0 & \text{if } t \in C_n
  \end{cases}
\]

where \( \alpha \) is so chosen that \( (\alpha - 2) F_K(\theta) > F_K(x) \). Choosing a suitable \( \lambda > 1 \) in \( x_\lambda = \lambda x + (1 - \lambda) q(x) \) satisfying \( F_K(x_\lambda) = F_K(u) \), we now apply Case I to the pair \((C_n, C_n')\) and observing that \( u - q(u) \) assumes its ess. sup norm on the set \( C_n' \), we obtain \( q(u) = q(x) \). The same way, we also obtain \( q(u) = q(y) \) and, consequently, \( q(x) = q(y) \).

On the other hand, if \( \mu (C_n) = 1 \) for all \( n \), then with \( C = \cap C_n \), we have \( \mu (C) = 1 \) and \( |x(t) - q(x)(t)| = F_K(x) \) and \( |y(t) - q(y)(t)| = F_K(y) \) for all \( t \in C \). Choose an \( E \subset C \) with \( 0 < \mu (E) < 1 \) and then define

\[
  u(t) = \begin{cases} 
  \alpha F_K(\theta) & \text{if } t \notin E \\
  0 & \text{if } t \in E
  \end{cases}
\]

where \( \alpha \) is so chosen that \( (\alpha - 2) F_K(\theta) > F_K(x) \). Again, an application of Case I to the pair \((E, E')\) leads to \( q(x) = q(y) \). Since \( x \) and \( y \) are arbitrary, \( K \) contains a single farthest point and, consequently, \( K \) must reduce to a single element.

The following is an analogue of Theorem 3 of Bosznay.

**Theorem 4**—Let \( H \) be a Hilbert space and let \( \{f_n \} \) be an orthonormal sequence in \( H \). Let \( Y = \{f_1, f_2, ..., f_n, ...\} \). Then, for all \( \epsilon > 0 \), there exists a \( \| \cdot \| \) norm in \( Y \) such that for all \( x \in Y \), \( (1 - \epsilon) \|x\| \leq \|x\| \leq (1 + \epsilon) \|x\| \), and in \( (Y, \| \cdot \|) \), every uniquely remotal set is a singleton.

**Proof**: The proof is similar to that of Theorem 3 of Bosznay and we give it just for completeness.

Denote

\[
Y_n = \{x \in \{f_1, f_2, ..., f_n\} : \|x\| = 1\},
\]

and

\[
G_1 = \{x \in \{f_1\} : \|x\| < 1\}.
\]
Suppose \( \{Y_1^1, Y_2^1, \ldots, Y_{k(1)}^1\} \) is an \( \epsilon/16 \)-net in \( G_1 \), and for \( n > 1 \), let \( G_n = \{x \in Y_n : d(x, \in Y_{n-1}) > \frac{\epsilon}{4} (1 - \frac{1}{2^n})\} \), and \( \{y_1^n, y_2^n, \ldots, y_{n(n)}^n\} \) be an \( \epsilon/4^{n+1} \)-net in \( G_n \). As \( Y_n \)'s are symmetric about the origin, so are the sets \( G_n \)'s and therefore, the set \( P = \bigcup_{n=1}^{\infty} \{y_i^n : 1 \leq i \leq k(n)\} \) can be assumed to be symmetric about the origin. Let \( \{x_n\}_{n \in \mathbb{N}} \) be an enumeration of \( P \).

Next, let \( x \in Y \) and \( \|x\| = 1 \). Then \( x \in Y_n \) for some \( n \) and \( d(y, G_{n+1}) = \frac{\epsilon}{4} (1 - \frac{1}{2^{n+1}}) \) for all \( y \in Y_n \). Considering the \( \epsilon/4^{n+1} \)-net of \( G_{n+1} \), an element \( y_i^{n+1} \in P \) can be chosen so that \( \|x - y_i^{n+1}\| \leq \frac{\epsilon}{4} + \frac{\epsilon}{4^{n+1}} < \epsilon \), and consequently

\[
\inf_n \|x - x_n\| < \epsilon. \tag{1}
\]

From the fact that \( Y_n \cap G_{n+1} = \emptyset \), and \( Y_n \cup G_{n+1} \subset Y_{n+1} \), it is easy to check that

\[
\min_{P \cap Y_{n+1}} \|x - P_{n+1}\| \leq \inf_{y \in P \cap Y'} \|x - y\|, \quad \forall m \geq n + 2.
\]

It now follows that \( \min_n \|x - x_n\| \) exists and, therefore, (1) reduces to

\[
\min_n \|x - x_n\| < \epsilon. \tag{2}
\]

In view of the identity

\[
< y, x_n > = \|y\| \left(1 - \frac{\|y\| - \|x_n\|^2}{2}\right) \tag{3}
\]

a norm \( \|\cdot\|_\epsilon \) can be defined on \( Y \) by the formula

\[
\|y\|_\epsilon = \max_n \max \{< y, x_n >, < y, - x_n >\}
\]

\[
= \max_n < y, x_n >, \text{ by the symmetry of } P.
\]

In view of (2) and (3) and the fact that \( \|x_n\| = 1 \), it is easy to check that \( \|\cdot\|_\epsilon \) is a norm with the desired property. The rest follows from Theorem 3 of Asplund⁴.

Now, we consider the farthest point problem in subalgebras of \( C(\Omega) \), where \( \Omega \) is a compact topological space. The same with \( \Omega \) compact and Hausdorff and the subalgebra separating and containing the constant functions has been considered by Bosznay⁵ in his Theorem 1.

**Theorem 5**—Every uniquely remotal set in a subalgebra \( \mathcal{A} \) of \( C(\Omega) \) is singleton.
Proof: Let $K$ be a uniquely remotal set in $\mathcal{A}$. We shall now say that an evaluation functional $\delta_{t_{0}} (t_{0} \in \Omega)$ corresponds to a farthest point $q (x)$ in $K$ if $\delta_{t_{0}} (x - q (x)) = \|x - q (x)\| = F_{K} (x)$ for some $x$ in $\mathcal{A}$.

Assume that the evaluation functionals $\delta_{t_{1}}$ and $\delta_{t_{2}}$ correspond to farthest points $q (x)$ and $q (y)$ respectively. As usual, there is loss of generality in assuming that $F_{K} (x) = F_{K} (y)$. We then obtain

$$x (t_{2}) - q (y) (t_{2}) \leq y (t_{2}) - q (y) (t_{2}) = F_{K} (y)$$

and

$$y (t_{1}) - q (x) (t_{1}) \leq x (t_{1}) - q (x) (t_{1}) = F_{K} (x).$$

Therefore,

$$x (t_{2}) \leq y (t_{2}) \text{ and } y (t_{1}) \leq x (t_{1}).$$

Equality of any of these two will lead to $q (x) = q (y)$. So we shall assume that $x (t_{2}) < y (t_{2})$ and $y (t_{1}) < x (t_{1})$. Now define a function $h$ by setting $h (t) = x (t) - y (t)$. Clearly, $h \in \mathcal{A}$, $h (t_{1}) > 0$ and $h (t_{2}) < 0$. Further define

$$\lambda (t) = \frac{h (t) - h (t_{2})}{h (t_{1}) - h (t_{2})} \cdot \frac{\delta_{t_{1}} (t)}{h (t_{1})}, \lambda \in \mathcal{A}$$

and

$$g (\lambda) = \frac{\lambda^{2} \left(m - \lambda^{2}\right)^{m-1}}{(m - 1)^{m-1}}, \|\lambda\| < \sqrt{m}$$

where $m$ is any integer greater than 3. Clearly $g_{\min} (\lambda) = 0$, $g_{\max} (\lambda) = 1$ and $g (\lambda) \in \mathcal{A}$. The function $z (t) = g (\lambda (t)) x (t) + (1 - g (\lambda (t))) y (t)$ is in $\mathcal{A}$ and has both $q (x)$ and $q (y)$ as farthest points. By the unique remotal property of $K$, it follows that $q (x) = q (y)$. Further, every evaluation functional corresponds to at the most one farthest point in $K$ (for example, put $t_{1} = t_{0}$ in the above). Thus the collection of all $\delta_{t}$’s ($t \in \Omega$) correspond to a single farthest point $q (x_{0})$ say, in $K$. Similar is the case for $- \delta_{t}$’s ($t \in \Omega$). If the latter farthest point is $q (y_{0})$ and if $q (x_{0}) \neq q (y_{0})$, then $(q (x_{0}) + q (y_{0}))/2$ will admit both $q (x_{0})$ and $q (y_{0})$ as farthest points which is a contradiction. This completes the proof.

We note that if $\Omega$ fails to be Hausdorff, then $C (\Omega)$, with $\Omega$ compact, may fail to contain any nonconstont continuous function. Nevertheless, we have the following:

Corollary—Every uniquely remotal set in $C (\Omega)$, where $\Omega$ is compact topological space, is a singleton.

In case $\Omega = [0, 1]$ the idea that whether or not the result of Theorem 5 could be extended to every subspace of $C (\Omega)$ is quite revealing. An affirmative answer would imply that every uniquely remotal set in a separable Banach space would be a singleton. This is due to the fact that every separable Banach space is congruent with a
subspace of $C [0, 1]$ (see Holmes[^4], p. 226) and, secondly, the solution to the farthest point problem remains unaffected in a congruent Banach space. This fact necessitates the study of the farthest point problem in certain special class of subspaces of $C [0, 1]$. To this end, we consider the following subspace of $C [0, 1]$. Let

$$Y = \{ x \in C [0, 1] : x^* (t) \text{ exists and is equal to zero }$$

for all $t \in [0, 1] \sim P$, where $P$ is the Cantor set). Obviously, Lebesgue's singular function is a typical element of the space $Y$. It can be easily checked that $Y$ is a complete normed linear space under the induced sup norm. The following theorem, now, generalizes the well-known result[^6] that any separable Banach space is congruent with a subspace of $C [0, 1]$.

**Theorem 6**—Any separable Banach space is congruent with a subspace of $Y$.

**Proof**: The proof is almost a reproduction of the same given in Holmes[^4] (p. 226). The continuous function has defined from $[0, 1]$ onto $U (X^*)$ can be seen to be linear on open intervals $(s_n, t_n)$ and, therefore, the inclusion map $i : x \rightarrow C U (X^*)$ followed by the congruence $T : C (U (X^*)) \rightarrow C [0, 1]$ takes $X$ into a subspace of $Y$ via the formula $(Tx)(t) = \langle x, h (t) \rangle \forall t \in [0, 1]$.

**Theorem 7**—Every uniquely remotal set $K$ in the space $Y$ is a singleton.

**Proof**: The proof follows from the fact that $Y$ is congruent to $C (P)$, where $P$ is the cantor set.

Our interest lies now in congruent images in $Y$ of separable Banach spaces. The solution to the farthest point problem in a separable Banach space would then reduce to that of an identical problem in a congruent subspace in $Y$. However, this remains an open problem.

**References**