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COMBINATORIAL PROOFS OF SOME ENUMERATION IDENTITIES

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We give a combinatorial proof of the following :

Theorem—The number of compositions of n is the same as the number of self-conjugate partitions with largest part equal to n.

Corollary—The number of n-reflected lattice paths equal twice the number of compositions of n. Some other enumeration identities are also proved.

1. INTRODUCTION

It is easily verified that the number of self-conjugate partitions of n with largest equal to m is the coefficient of $t^m q^n$ in

$$\sum_{m=0}^{\infty} \frac{t^m q^{m^2}}{(1 - tq^2)(1-tq^4) \dots (1 - tq^{2m})} .$$

Hence the generating function for all self-conjugate partitions with largest part equal to m is what we get when we set $q = 1$:

$$\sum_{m=0}^{\infty} \frac{t^m}{(1 - t)^m} = \frac{1 - t}{1 - 2t}$$

or

$$\sum_{m=1}^{\infty} \frac{t^m}{(1-t)^m} = \frac{t}{1 - 2t} . \qquad ...(1.1)$$

On the other hand, if $C_m (t)$ is the enumerating generating function for compositions with exactly m parts, then

$$C_m (t) = (t + t^2 + t^3 + \dots)^m = t^m (1 - t)^{-m}. \qquad ...(1.2)$$

And so the generating function with no restriction on the number or size of parts is

$$C (t) = \sum_{m=1}^{\infty} C_m (t) = \frac{t}{1 - 2t} . \qquad ...(1.3)$$

A comparison of (1.1) and (1.3) leads us to our main theorem.

Theorem—The number of composition of n is the same as the number of self-conjugate partitions with largest part equal to n.

The exact meaning of the theorem should be clear from the following example : There are eight compositions of 4, viz.

$$4,\ 31,\ 13,\ 2^2,\ 21^2,\ 121,\ 1^22,\ 1^4.$$

There are also eight self-conjugate partitions with largest part 4, viz.

$$41^3,\ 421^2,\ 4321,\ 43^21,\ 4^22^2,\ 4^232,\ 4^33,\ 4^4.$$

This paper is centered around a combinatorial proof of the main theorem. The n-reflected lattice paths of the corollary were recently studied by Agarwal and Andrews[1], and are defined as follows :

Definition—A lattice path from (0, 0) to (n, n) is said to be n-reflected if for each (x, y) in the path $(n - y, n - x)$ is also in the path. For example, there are four 2-reflected lattice paths, viz.,

(0, 0), (1, 0), (2, 0), (2, 1), (2, 2); (0, 0), (1, 0) (1, 1), (2, 1), (2,2);

(0, 0), (0, 1), (1, 1), (1, 2), (2,2); (0, 0), (0, 1), (0, 2), (1, 2), (2,2).

In the next section we shall prove some subsidiary theorems which are required in proving our main results combinatorially.

2. Subsidiary Theorems

Theorem 2.1—The number of compositions of n with exactly m parts equals the number of partitions into m distinct parts with largest part n.

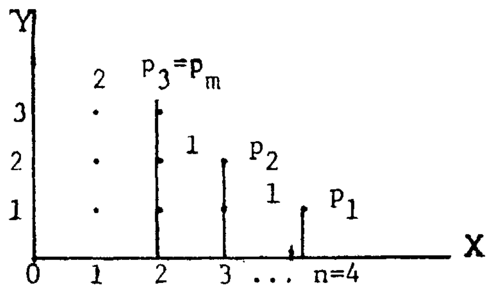
First proof (*by generating function*)—It is well known that the coefficient of $t^n q^N$ in

$$\frac{t^m q^{m(m+1)/2}}{(1-tq)(1-tq^2)\ldots(1-tq^m)}$$

is the number of the partitions of N into m distinct parts with largest part n. Hence the generating function for all the partitions into m distinct parts with largest part n is what we get when we set $q = 1$, viz.,

$$\frac{t^m}{(1-t)^m},$$

and the theorem follows immediately by noting that this function also enumerates the compositions with exactly m parts [see eqn. (1.2)].



Second proof (combinatorial) —Let be the graph of a partition into m distinct parts with largest part n (In the above graph $n = 4$, $m = 3$; also note that the X-axis is drawn one unit of length below the last row and the Y-axis one unit of length to the left of the first column). We draw vertical lines from the corner point of each part and measure the distance of each line from its preceding one taking.

Y-axis also into consideration. We see that these distances give rise to a composition of n into exactly m parts. The correspondence being one-one the theorem is proved.

$4 + 3 + 2 \longrightarrow 211$

$4 + 3 + 1 \longrightarrow 121$

$4 + 2 + 1 \longrightarrow 112$

Theorem 2.2—The number of compositions of n is the same as the number of partitions into at most n distinct part with largest part n.

PROOF : Let $P(n, m)$ denote the number of partitions into m distinct parts with largest part n, and $C_{n,m}$ the number of compositions of n with exactly m parts. Then by Theorem 2.1, we have

$$C_{n,m} = P(n, m). \qquad \text{...(2.1)}$$

Equation (2.1) implies

$$\sum_{m=1}^{n} C_{n,m} = \sum_{m=1}^{n} P(n, m),$$

and the theorem follows immediately.

The one-to-one correspondence established to prove Theorem 2.1 holds good here too.

$4 \longleftrightarrow 4$

$4 + 3 \longleftrightarrow 3,1$

$4 + 2 \longleftrightarrow 2,2$

4 + 1	←→ 1,3
4 + 3 + 2	←→ 2, 1, 1
4 + 3 + 1	←→ 1, 2, 1
4 + 2 + 1	←→ 1, 1, 2
4 + 3 + 2 + 1	←→ 1, 1, 1, 1

Now we shall establish a natural bijection for the following :

Theorem 2.3—The number of self-conjugate partitions with largest part n is the same as the number of partitions into at most n distinct parts with largest part n.

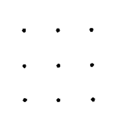
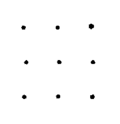
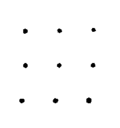
We first define a k-bend.

Definition —We call a right-bend ......, a k-bend if the number of dots in first row and first column are both equal to k. Thus by 1-bend we mean single dot., by 2-bend ... by 3-bend ..., etc.

Proof of the Theorem 2.3—Let $\pi = a_1 + a_2 + \ldots + a_r$ $(\eta = a_1 > a_2 > \ldots > a_r)$ be a partition

We consider a graph which consists of r successive bends viz., a_1-bend, a_2-bend,. ...a_r-bend. We see immediately that this graph represents a self-conjugate partition with largest part equal to n. The correspondence being one-to-one, the theorem is proved.

Example—Consider the case $n = 4$.

Partitions into at most 4 distinct parts with largest part 4	Mapping	Self-conjugate partition with largest part 4
4 + 3 + 2 + 1	(4-bend) + (3-bend) + (2-bend) + (1-bend). ←——→	
4 + 3 + 2	(4-bend) + (3-bend) + (2-bend) ←——→	
4 + 3 + 1	(4-bend) + (3-bend) + (1-bend) ←——→	

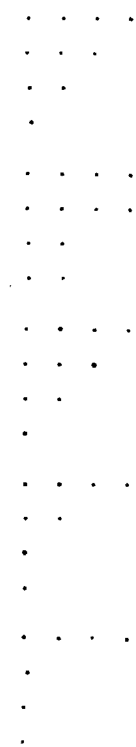
4 + 2 + 1 (4-bend) + (2-bend) + (1-bend)

4 + 3 (4-bend) + (3-bend)

4 + 2 (4-bend) + (2-bend)

4 + 1 (4-bend) + (1-bend)

4 4-bend



3. Proofs of the Main Results

The bijections established to prove Theorems 2.2 and 2.3 together give rise a natural one-one onto mapping for our main theorem.

A bijection for the corollary can easily be obtained once we recall the following result due to Agarwal and Andrews[1].

Theorem 2.4—The n-reflected lattice paths are in one-to-one correspondence with the self-conjugate partitions with largest part $\leqq n$.

4. Conclusion

Since a subset of $\{1, 2, ..., n\}$ is also a partition into at most n distinct parts with largest part $\leqslant n$, the discussion of the preceding sections was a natural way to look at the following :

Theorem 4.1—Let A_n be the number of compositions of all integers $\leqq n$. B_n the number of self-conjugate partitions with largest part $\leqq n$. C_n the number of n-reflected lattice paths and D_n the number of subsets of $\{1, 2, ..., n\}$.

Then

$$A_n = B_n = C_n = D_n = 2^n.$$

Reference

1. A. K. Agarwal, and G. E. Andrews, *J. Stat. Pl. Innf.* **14**, (1986), 5-14.