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ALGEBRAS WITH A UNITY COMMUTATOR*

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A pair of elements in a linear associative algebra (or a ring) is called a 'Heisenberg pair' if their commutator is unity. Necessary conditions or sufficient conditions are given for existence of a Heisenberg pair in various rings and algebras. Emphasis is on rings of endomorphisms on modules and on rings of matrices. The situation is completely analyzed for free modules. Consequences of an algebra containing a Heisenberg pair are developed. The paper lists some open questions and a conjecture.

1. INTRODUCTION

Questions concerning the existence and nature of linear operators whose commutator is a scalar multiple of the identity can be traced back to the early days of modern quantum mechanics[7]. Even in the special setting appropriate to quantum mechanics, the existence question was not satisfactorily answered until 1947[10]. Since then the study of commutators of linear operators on a Hilbert space has attracted considerable attention. (Halmos[5] and Putnam[8] discuss this in detail and each provide extensive references.) This paper addresses the collateral question of the nature of morphisms whose commutator is the identity. More specifically, if R is a ring and M is an R-module, then a pair α, β of R-endomorphisms on M is called a 'Heisenberg pair' if $\alpha\beta - \beta\alpha = I_M$. (Herein all rings have unity : modules are unital left modules ; subrings contain the same unity element as the ring ; and ring homomorphisms take unity into unity, unless otherwise noted.) Necessary and sufficient conditions for a free module to have a Heisenberg pair are given. It is useful—and of interest of itself—to consider Heisenberg pairs in the setting of algebras and rings. A ring (algebra) A is said to contain a Heisenberg pair if $a, b \in A$ such that the commutator $[a,b] = ab - ba$, is unity. This paper addresses the question of when a ring or algebra can or cannot contain a Heisenberg pair, with special consideration given to rings of endomorphisms or matrices. (Shoda[9] and Albert and Muckenhaupt[1] have considered commutators of matrices over fields.)

For a ring R, char R denotes the additive order of the unity of the ring ; $C(R)$ denotes the center of the ring; and $M_n(R)$ is the full ring of n by n matrices over R. If

*This paper is dedicated to Professor E. R. Keown.

M is an R-module, then $\mathrm{End}_R M$ is the ring of all R-endomorphisms on M and I_M is the identity mapping on M. An 'algebra' will mean a unital associative linear algebra over a commutative ring.

By way of explanation for the choice of the phrase "Heisenberg pair," it was picked up from correspondence and conversation with P. R. Halmos. He used it in the context of linear operators, but seeing as how it reveals the origins of this study, which now roams through rings, algebras, and modules, it seems appropriate to use it also in the more general context.

2. Heisenberg Pairs on Free Modules

A free module is said to have 'dimension' if every free basis of the module has the same cardinality; if that cardinal is n (finite or infinite), the module is said to have 'dimension' n. (See Leavitt.[6]) A ring R is said to be 'dimensional if every free R-module has dimension. (Cohn[2] uses the phrase 'invariant basis number.) If M is a free module which is not finitely generated, then M has dimension[4]. The class of dimensional rings includes all commutative rings[3] and all subrings of left or right Noetherian rings[2]. In this section conditions are given which guarantee the existence of a Heisenberg pair on a free module which has dimension.

Proposition 2.1—Let F be a free R-module. If either of the following hold, then there is a Heisenberg pair on F:

(1) F is not finitely generated, or

(2) F has finite dimension $n \geqslant 2$ and char R divides n.

Proof : Let B be a free basis for F. If B is infinite, write B as the disjoint union of sets $B_j = \{b_{ji} : i = 0; 2, \ldots\}$. Define functions α and β on B via :

$$\alpha b_{j0} = 0,\ \alpha b_{ji} = i b_{j,i-1}, \text{ for } i \geqslant 1.$$

$$\beta b_{ji} = b_{j,i+1}, \text{ for } i \geqslant 0.$$

Then $\alpha\beta - \beta\alpha = I_B$. Extend α and β to R-endomorphisms on F to obtain the desired Heisenberg pair.

If $|B| = n$, then write B as the disjoint union of sets $B_j = \{b_{ji} : i = 0, 1, \ldots, m - 1\}$, where $m = \text{char } R$. Then define α and β via:

$$\alpha b_{j0} = 0,\ \alpha b_{ji} = i b_{j,i-1}, \text{ for } i = 1, \ldots, m - 1.$$

$$\beta b_{ji} = b_{j,i+1}, \text{ for } i = 1, \ldots, m - 2 \text{ and } \beta b_{j,m-1} = 0.$$

Then extend α and β to R-endomorphisms on F and the desired Heisenberg pair is obtained.

If char R does not divide the dimension of F in the finite dimensional case, the given α and β will not yield $\alpha\beta - \beta\alpha = 1$, since $(\alpha\beta - \beta\alpha)\, b_{j,m-1} = (1 - m) b_{j,m-1}$.

Variations on the scheme given in the proof of Proposition 2. 1 yield many different Heisenberg pairs. In particular, if $\gamma \in \mathrm{End}_R F$ is invertible, then $\gamma\alpha\gamma^{-1}$, $\gamma\beta\gamma^{-1}$ is a Heisenberg pair if and only if α, β is.

As the next result shows, the dimension of F being a multiple of char R is necessary.

Proposition 2.2—Let R be a commutative ring and F a free R-module of finite dimension n. If there is a Heisenberg pair on F, then char R divides n.

PROOF : Since $\mathrm{End}_R F$ is ring isomorphic to $M_n(R)$, there is a Heisenberg pair A, B in $M_n(F)$. However, the trace of $[A, B]$ is zero while the trace of $I \in M_n(F)$ is $n.\ 1$, which is zero only if char R divides n.

Corollary 2.3—If R has characteristic zero, then a finite dimensional free module over R cannot have a Heisenberg pair.

The argument in Proposition 2. 2 realise on the trace property; tr AB = tr BA, which does not hold in general for matrices over noncommutative rings. In the next section a way around this obstacle will be found.

3. HEISENBERG PAIRS IN RINGS AND ALGEBRAS

Proposition 3.1—Let A be an algebra over a commutative ring R such that A is a finite dimensional free module over R. If A contains a Heisenberg pair, then char R divides $\dim_R A$.

PROOF : For each $a \in A$, define $\phi_a(x) = ax$, the left multiplication mapping induced by a. Then $a \to \phi_a$ gives an R-algebra monomorhism from A into $\mathrm{End}_R A$; this mapping takes $I \in A$ into I_A. The existence of a Heisenberg pair on A then yields a Heisenberg pair in $\mathrm{End}_R A$, which in turn forces : char R divides $\dim_R A$.

Proposition 3.2—Let R be a subring of a simple ring S with $0 < \dim_{C(S)} S = k < \infty$. If char S does not divide $n.\ k$, then $M_n(R)$ does not contain a Heisnberg pair.

PROOF : As in the proof of Proposition 3.1, there is a ring monomorphism from S into $\mathrm{End}_{C(S)} S$ and hence into $M_k(C(S))$. Using this yields a ring monomorphism from $M_n(S)$ into $M_{nk}(C(S))$; unity goes into unity with each morphism. Since char $C(S)$ = char S does not divide $n.\ k.$ there is no Heisenberg pair in $M_{nk}(C(S))$ and hence none in $M_n(S)$. Consequently, $M_n(R)$ cannot contain a Heisenberg pair.

Corollary 3.3—Let S be a simple ring which is finite dimensional over its center. If char S does not divide $\dim_{C(S)} S$, then S does not contain a Heisenberg pair.

Proposition 3.2 is of particular interest if S is a skewfield. One consequence of this and of Proposition 2.1 is the following extension of a venerable piece of linear algebra folklore.

Proposition 3.4—Let V be a vector space over a skewfield K such that $\dim_{C(K)} K = m < \infty$. Then the following are equivalent :

(a) There is a Heisenberg pair either in $\mathrm{End}_K V$ or in $M_m(K)$.

(b) Either (i) $\dim_K V = \infty$ or (ii) $\dim_K V = n < \infty$ and char K divides mn.

PROOF : The case where $\dim_K V = 1$ is trivial, so take $n > 1$ throughout the proof.

If there is a Heisenberg pair in $\mathrm{End}_K V$ and $\dim_K V = n < \infty$, then there is a Heisenberg pair in $M_n(K)$. The center of $M_n(K)$ is ring isomorphic to $C(K)$. Using $S = M_n(K)$ as the simple ring in Proposition 3.2 and $R = \{\lambda I_n : \lambda \in K\}$, since $\dim_{C(S)} S = n^3 m$, we have char S divides $n^3 m$. But char S = char K, which in this case is a prime, leading to char K divides either n or m.

If $M_m(K)$ contains a Heisenberg pair, then using $S = K = R$ in Proposition 3.2 yields char K divides m^2 and hence divides m.

If either $\dim_K V = \infty$ or $\dim_K V = n < \infty$ and char K divides n, then Proposi- 2.1 immediately gives a Heisenberg pair in $\mathrm{End}_K V$. If char K divides m, then free K-module K^m satisfies Proposition 2.1 (2) and hence $\mathrm{End}_K K^m \cong M_m(K)$ contains a Heisenberg pair.

Given a ring (or algebra) which contains a Heisenberg pair, it is easy to form new which also have this property. Let Ω be the class of all rings (algebras) which contain a Heisenberg pair.

Then Ω is closed under the formation of :

(i) factor rings,

(ii) finite direct sums.

(iii) direct products,

(iv) full matrix rings.

Proposition 3.5—Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of R-algebras, where each is also a free R-module, R a commutative ring of characteristic zero, and f and g be epimorphisms. If B contains a Heisenberg pair, then each of A, B, C is infinite dimensional over R.

PROOF : Proposition 2.2 yields that, viewed as a free R-module, B must have infinite dimension. So A and C must be infinite dimensional over R.

Corollary 3.6—Let M be an infinite dimensional free R-module, where R is a commutative ring of characteristic zero. The R-algebra $\mathrm{End}_R M$ has no finite dimensional (non-zero) homomorphic image.

Immediately from Proposition 3.1 it is seen that if A is an algebra over a commutative ring R, char $R = 0$, and A contains a Heisenberg pair, then $\dim_R A$ is infinite. In the case where R is a field, there is an easy alternate proof of this result which makes on use of trace.

ALTLRNATE PROOF : If $a, b \in A$ such that $[a, b] = 1$, then $[a^{n+1}, b] = (n + 1)a^n$. Use this and induction on n to show $\{a_j : j = 0, 1, 2, \ldots, n\}$ is linearly independent for each n. (Both char $R = 0$ and R a field play a crucial role in this argument.)

4. QUESTIONS AND CONJECTURES

Question 4.1—For a given ring R, which R-modules will have a Heisenberg pair of R-endomorphisms?

For free modules this question was answered in Section 2. Considering the intimate connection between free modules and projective modules it does not seem unreasonable that similar results hold for projective modules. However, the obvious attack does not work.

Conjucture 4.2—If P is a projective R-module which is not finitely gererated, then there is a Heisenberg pair in $\mathrm{End}_R P$.

Another approach to answering Question 4.1 is to consider the exact sequence of R-modules : $F \to M \to 0$, where F is free, and look for a ring homomorphism from $\mathrm{End}_R F$ onto $\mathrm{End}_R M$, a situation which does not happen in general, but which can occur.

Question 4.3—In the setting above, when is $\mathrm{End}_R M$ a ring homomorphic image of $\mathrm{End}_R F$?

For skewfield K every K-module which is not finitely generated has a Heisenberg pair. It is natural to ask :

Question 4.4—Which rings have the property that every module which is not finitely generated has a Heisenberg pair?

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