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APPROXIMATION OF A FUNCTION BY THE $F(a, q)$ TRANSFORM OF ITS FOURIER SERIES

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Results on the order of approximation of a 2π-periodic continuous function by the Euler or Taylor means of the sequence of partial sums of its Fourier series are extended to general class of $F(a, q)$ transform of which these and other transforms known as Kreisverfahran are special cases.

1. INTRODUCTION

Let $C[0, 2\pi]$ denote the class of all continuous 2π periodic functions. If $f \in C[0, 2\pi]$, ω_f denotes its modulus of continuity. Let the Fourier series associated with $f \in C[0, 2\pi]$ at x be

$$a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \quad ...(1.1)$$

As usual let us write

$$\phi_x(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}. \quad ...(1.2)$$

The kth partial sum $s_k(x)$ of the Fourier series is given by

$$s_k(x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \frac{\phi_x(t)}{\sin \frac{1}{2}t} \sin(k + \tfrac{1}{2})\, t\, dt. \quad ...(1.3)$$

The family $F(a, q)$ of summability methods was introduced by Meir[6]. An $F(a, q)$-transform of the sequence $\{s_k(x)\}$, the sequence of partial sums of the Fourier series of $f \in C[0, 2\pi]$ at x, is

$$\sigma_p(x) = \sigma_p(a, q, f, x) = \sum_{k=0}^{\infty} c_k(p)\, s_k(x),\ c_k(p) \geqslant 0 \quad ...(1.4)$$

where, for $g(q, k)$ defined by

$$g(q, k) = \sqrt{a/\pi q}\, \exp\{-aq^{-1}(k-q)^2\},\ a > 0,\ q = q(p) \quad ...(1.5)$$

which is a positive non-decreasing function of a continuous or discrete parameter p, which tends to infinity as $p \to \infty$, and for some fixed γ, $\frac{1}{2} < \gamma < \frac{2}{3}$,

$$c_k(p) = q(q, k)\left\{1 + O\left(\frac{|k-q|+1}{q}\right) + O\left(\frac{|k-q|^3}{q^2}\right)\right\} \quad \ldots(1.6)$$

as $p \to \infty$ uniformly in k for $|k - q| \leqslant q^\gamma$, while

$$\sum_{|k-q| > q^\gamma} (k+1)\, c_k(p) = O\{\exp(-q^\mu)\} \quad \ldots(1.7)$$

as $p \to \infty$, for some positive number μ independent of p.

The family $F(a, q)$ is known[5] to contain the summability methods of generalised Borel, Euler, Taylor, S_β (defined explicitly later) and Valiron. It is known[5] that

$$\sum_{k=0}^{\infty} c_k(p) = 1 + O(q^{-1/2}). \quad \ldots(1.8)$$

The summability methods of Euler, Taylor, S_β and Borel satisfy (1,8) in the stronger form

$$\sum_{k=0}^{\infty} c_k(p) = 1. \quad \ldots(1.9)$$

In what follows ω is a positive non-decreasing function such that

$\omega(t)/t^{1/2}$ is non-increasing function of $t \in [0, \pi]$. ...(1.10)

As a consequence of this condition (1.10) on ω, for $\lambda > 1$, we get

$$\omega(\lambda t) \leqslant \gamma\lambda\, \omega(t)\ (t > 0). \quad \ldots(1.11)$$

In the sequel in order relations involving q it is to be understood that $p \to \infty$.

Chui and Holland[1] proved that the order of approximation of functions in the class Lip α by either Euler-$(E, 1)$ means or Taylor means of Fourier series can be re-reduced to Jackson order provided, in each case, a suitable integrability condition is imposed upon $\phi_x(t)$. Xie[8] extended this result to Euler-(E, q) $(q > 0$ means in the context of continuous 2π periodic functions Chui *et al.*[3] (p.373, Corollary 5.15) seem to have obtained the analogue of the above result for the Borel method, but the details do not seem to have been published. We append the details pertaining to the analogue for the S_β method in Thsorem 2. We extend this result to the family $F(a, q)$ in the context of the class Lip α, but with the restriction $0 < \alpha \leqslant \frac{1}{2}$.

We prove the following results

Theorem 1—Let n be the integral part of $q = q(p)$. Set $m = n + 1$. Let $f \in C[0, 2\pi]$ such that $\omega_f \geqslant \omega$, and satisfy the condition

$$\int_{u(m)}^{v(m)} \frac{| \phi_x(t) - \phi_x(t + u(m)) |}{t} \exp(-mt^2/4a)\, dt = O\left(\omega\left(\frac{1}{m}\right)\right) \quad (1.12)$$

uniformly in x. Then

$$\max_{0 \leqslant x \leqslant 2\pi} | \sigma_p(x) - f(x) | = O\left(\omega\left(\frac{1}{m}\right)\right) \quad ...(1.13)$$

where

$$u(m) = \pi/m + \tfrac{1}{2},\ v(m) = [u(m)]^{\eta},\ 0 < \eta < \tfrac{1}{2}. \quad ...(1.14)$$

In view of (1.10), Theorem 1 immediately yields the

Corollary--If $\omega(t) = t^{\alpha}$, $0 < \alpha \leqslant \frac{1}{2}$ and $f \in C[0, 2\pi]$ such that $\omega_f \leqslant \omega$ and satisfy the condition (1.12) uniformly in x, then

$$\max_{0 \leqslant x \leqslant 2\pi} | \sigma_p(x) - f(x) | = O(m^{-\alpha}). \quad ...(1.15)$$

This corollary with the restriction on α weakened to $0 < \alpha < 1$, is due to Chui and Holland[1] for the methods of Euler-$(E, 1)$ and Taylor and due to Xie[8] for the method of Euler (E, q) $(q > 0)$. The following Theorem 2 is the analogue of these results for the method S_β $(0 < \beta < 1)$ for which the transform $\sigma_p \equiv S_\beta^p$ of the sequence $\{s_k\}$ is defined with

$$c_k(p) = (1 - \beta)^{p+1} \binom{p + k}{k} \beta^k,\ k, p = 0, 1, 2, \dots.$$

Theorem 2—Let $f \in \text{Lip}\,\alpha$, $0 < \alpha < 1$. Let S_β^p $(p = 0, 1, 2, \dots)$ denote the pth S_β-mean of the Fourier series.

If

$$\int_{a(p)}^{b(p)} \frac{| \phi_x(t) - \phi_x(t + a(p)) |}{t} \exp\{-\tfrac{1}{2} p\beta (t/1-\beta)^2\}\, dt$$

$$= O(p^{-\alpha}), \quad ...(1.16)$$

where

$$a(p) = \frac{\pi(1 - \beta)}{(p + 1)\beta},\ b(p) = [a(p)]^{\delta},\ \frac{1 + \alpha}{3 + \alpha} < \delta < \tfrac{1}{2}, \text{ then}$$

$$\max_{0 \leqslant x \leqslant 2\pi} | S_\beta^p(x) - f(x) | = O(p^{-\alpha}). \quad ...(1.17)$$

2. Preliminary Results

To prove the results in section 1 we need the following lemmas.

Lemma 1—If $q = q(p)$ is an integer valued function of p, then, for $\frac{1}{2} < \gamma < \frac{2}{3}$ we have

$$\int_0^{\pi} \frac{\phi_x(t)}{\sin t/2} \sum_{|k-q| \leqslant q^{\gamma}} g(q, k) \sin (k + \tfrac{1}{2}) t \, dt$$

$$= \int_0^{\pi} \frac{\phi_x(t)}{\sin t/2} \exp(-qt^2/4a) \sin (q + \tfrac{1}{2}) t \, dt$$

$$+ O(q^{3\gamma/2}. \exp(-aq^{2\gamma-1})). \qquad ...(2.1)$$

Proof : Following the proof of Lemma 3.2 of Ikeno[4] we have

$$\sum_{|k-q| \leqslant q^{\gamma}} g(q, k) \sin (k + \tfrac{1}{2}) t \, dt$$

$$= \exp(-qt^2/4a) \sin (q + \tfrac{1}{2}) t$$

$$+ O\{ \sum_{|r> | q^{\gamma}} \sqrt{a/\pi q} \exp(-ar^2/q) \,|\sin (r + q + \tfrac{1}{2}) t|\},$$

where $r = k - q$. Now we estimate the error term :

$$\sum_{|r| > q^{\gamma}} \sqrt{a/\pi q} \exp(-ar^2/q) \,|\sin(r + q + \tfrac{1}{2}) t|$$

$$\leqslant \sum_{|r| > q^{\gamma}} \sqrt{a/\pi q} \exp(-ar^2/q) (|r| + q + \tfrac{1}{2}) t$$

$$= 2 \sqrt{a/\pi q}\, t \{ \sum_{r > q^{\gamma}} r \exp(-ar^2/q)$$

$$+ (q + \tfrac{1}{2}) \sum_{r > q^{\gamma}} \exp(-ar^2/q)\}.$$

Now, for large q,

$$\sum_{r > q^{\gamma}} r \exp(-ar^2/q) \leqslant q^{\gamma} \exp(-aq^{2\gamma-1}) + \int_{q^{\gamma}}^{\infty} y \exp(-ay^2/q) \, dy$$

$$= q^{\gamma} \exp(-aq^{2\gamma-1}\} + \frac{q}{2a} \int_{aq^{2\gamma-1}}^{\infty} e^{-z} \, dz.$$

Using the fact that, for real θ

$$\int_{\lambda}^{\infty} z^{\theta} e^{-z} \, dz = O(\lambda^{\theta} c^{-\lambda}), \text{ as } \lambda \to \infty \qquad ...(2.2)$$

we get

$$\sum_{r>q^{\gamma}} r \exp(-ar^2/q) \leqslant q^{\gamma} \exp(-aq^{2\gamma-1}) + O(q \exp(-aq^{2\gamma-1})).$$

Similarly, we can prove

$$\sum_{r>q^{\gamma}} \exp(-ar^2/q) \leqslant \exp(-aq^{2\gamma-1}) + O(q^{1-\gamma} \exp(-aq^{2\gamma-1})).$$

Thus

$$\sum_{|r|>q^{\gamma}} \sqrt{a/\pi q} \exp(-ar^2/q) \,|\sin(r + q + \tfrac{1}{2}) t|$$

$$= O(\sqrt{q}\, t \exp(-aq^{2\gamma-1})) + O(q^{(3-2\gamma/2)}\, t \exp(-aq^{2\gamma-1})).$$

Hence the lemma follows in view of boundedness of $\phi_x(t)$ and since $\sin\left(\frac{t}{2}\right) > \left(\frac{t}{\pi}\right)$ $(0 < t < \pi)$.

Lemma 2 – Let $f \in C[0, 2\pi]$ such that $\omega_f \leqslant \omega$, where ω is as in section 1, and let condition (1.12) be satisfied. Then

$$\int_0^{\pi} \frac{\phi_x(t)}{\sin t/2} \exp(-mt^2/4a) \sin(m + \tfrac{1}{2}) t\, dt = O\, \omega(\frac{1}{m})).$$

PROOG : If u and v are as defined in (1.14) we write

$$\int_0^{\pi} \frac{\phi_x(t)}{\sin t/2} \exp(-mt^2/4a) \sin(m + \tfrac{1}{2}) t\, dt$$

$$= \left(\int_0^{u(m)} + \int_{u(m)}^{v(m)} + \int_{v(m)}^{\pi}\right) \frac{\phi_x(t)}{\sin t/2} \exp(-mt^2/4a) \sin(m + \tfrac{1}{2}) t\, dt$$

$$= J_1 + J_2 + J_3, \text{ say.} \qquad \ldots(2.3)$$

Now

$$|J_1| \leqslant \pi \int_0^{u(m)} \frac{|\phi_x(t)|}{t} \exp(-mt^2/4a) \,|\sin(m + \tfrac{1}{2}) t|\, dt$$

$$\leqslant \pi \int_0^{u(m)} \frac{\omega(t)}{(t)} (m + \tfrac{1}{2}) t\, dt$$

$$\leqslant \pi^2 \omega(u(m)) = O(\omega(\frac{1}{m})). \qquad \ldots(2.4)$$

Using (2.2) we estimate J_3:

$$|J_3| \leqslant \pi \int_{\nu(m)}^{\pi} \frac{|\phi_x(t)|}{t} \exp(-mt^2/4a)\, dt$$

$$= O\{\int_{\nu(m)}^{\infty} 1/t \exp(-mt^2/4a)\, dt\}$$

$$= O\left\{\int_{\frac{m[\nu(m)]^2}{4a}}^{\infty} \frac{1}{z} e^{-z}\, dz\right\}$$

$$= O\{m^{2\eta-1} \exp(-\frac{\pi^{2\eta}}{4a} m^{1-2\eta})\}. \qquad ...(2.5)$$

Now, for $\alpha > 0$, $A > 0$, λ and δ any real constants we have

$$m^{\lambda} \exp(-Am^{\alpha}) = O(m^{\delta}). \qquad ...(2.6)$$

Also, by the condition (1.10) on ω,

$$q^{-1/2},\ m^{-1/2} = O(\omega(\frac{1}{m})). \qquad ...(2.7)$$

Thus it is enough to show that

$$|J_2| = O(\omega(\frac{1}{m})).$$

Since

$$\operatorname{cosec} t - \frac{1}{t} = O(t)$$

with (1.10), we have

$$J_2 = \int_{u(m)}^{v(m)} \frac{\phi_x(t)}{\sin \frac{1}{2}t} \exp(-mt^2/4a) \sin(m + \tfrac{1}{2}t)\, dt$$

$$= 2\int_{u(m)}^{v(m)} \frac{\phi_x(t)}{t} \exp(-mt^2/4a) \sin(m + \tfrac{1}{2})t\, dt$$

$$+ \int_{u(m)}^{v(m)} \phi_x(t)(\operatorname{cosec} \frac{t}{2} - \frac{2}{t}) \exp(-mt^2/4a) \sin(m+\tfrac{1}{2})t\, dt$$

(*equation continued on p.* 375)

$$= 2 \int_{u(m)}^{v(m)} \frac{\phi_x(t)}{t} \exp(-mt^2/4a) \sin(m + \tfrac{1}{2}) t \, dt + O(\omega \frac{1}{m}))$$

$$= J_4 + O(\omega(\frac{1}{m})), \text{ say.} \qquad \ldots(2.8)$$

We shall write J_4 as follows :

$$J_4 = \int_{u(m)}^{v(m)} \frac{\phi_x(t)}{t} \exp(-mt^2/4a) \sin(m + \tfrac{1}{2}) t \, dt$$

$$- \int_0^{v(m)-u(m)} \frac{\phi_x(t + u(m))}{t + u(m)} \exp(-m(t + u(m))^2/4a)$$

$$\times \sin(m + \tfrac{1}{2}) t \, dt$$

$$= \int_{u(m)}^{v(m)} \frac{\phi_x(t) - \phi_x(t + u(m))}{t} \exp(-mt^2/4a) \sin(m + \tfrac{1}{2}) t \, dt$$

$$+ \int_{u(m)}^{v(m)} \frac{\phi_x(t + u(m))}{t} [\exp(-mt^2/4a)$$

$$- \exp(-m(t + u(m))^2/4a)] \sin(m + \tfrac{1}{2}) t \, dt$$

$$+ \int_{u(m)}^{v(m)} \phi_x(t + u(m)) \exp(-m(t + u(m))^2/4a) \left[\frac{1}{t} \right.$$

$$\left. - \frac{1}{t + u(m)} \right] \sin(m + \tfrac{1}{2}) t \, dt$$

$$- \int_0^{u(m)} \frac{\phi_x(t + u(m))}{t + u(m)} \exp(-m(t + u(m))^2/4a) \sin(m + \tfrac{1}{2}) t \, dt$$

$$+ \int_{v(m)-uu(m)}^{v(m)} \frac{\phi_x(t + u(m))}{t + u(m)} \exp(-m(t + u(m))^2/4a) \sin$$

$$(m + \tfrac{1}{2}) t \, dt = I_1 + I_2 + I_3 + I_4 + I_5.$$

By hypothesis

$$I_1 = O\ (\omega\ (\frac{1}{m}\ )). \qquad \ldots(2.9)$$

By mean value theorem

$$\exp\ (-mt^2/4a)\ -\exp\ (-m(t+u(m))^2/4a) = \frac{2u\ (m)m\theta}{4a} \exp\ (-m\theta^2/4a)$$

for some θ such that $t < \theta < t + u\ (m) < 2t$. Hence

$$\exp\ (-mt^2/4a)\ -\exp\ (-m\ (t + u\ (m))^2/4a) = O\ (t \exp\ (-mt^2/4a)).$$

Thus

$$|\ I_2\ | = O\ \{\int_{u(m)}^{v(m)} |\ \phi_x\ (t + u\ (m))\ |\ \exp\ -mt^2/4a)\ dt\}.$$

By (1.10) and (1.11), we have

$$|\ I_2\ |\ = O\ \{\sqrt{m}\ \omega\ (\frac{1}{m}\ ) \int_{n(m)}^{v(m)} t^{1/2} \exp\ (-mt^2\ /4a)\ dt\}$$

$$= O\ (\omega\ (\ \frac{1}{m})). \qquad \ldots(2.10)$$

Again, by (1.10) and (1.11) we have

$$|\ I_3\ | \leqslant u\ (m) \int_{u^{(m)}}^{v(m)} \frac{|\ \phi_x\ (t + u\ (m))\ |}{t\ (t + u\ (m))}\ dt$$

$$= O\ \{u\ (m) \sqrt{m}\ \omega\ (\frac{1}{m}) \int_{u(m)}^{v(m)} \frac{1}{t\ (t + u\ (m))^{1/2}}\ dt\}$$

$$= O\ (\omega\ (\frac{1}{m}\ )). \qquad \ldots(2.11)$$

Also,

$$|\ I_4\ | \leqslant \int_0^{u(m)} \frac{|\ \phi_x\ (t + u\ (m))\ |}{t + u\ (m)}\ |\ \sin\ (m + \tfrac{1}{2})\ t\ |\ dt$$

$$\leqslant \int_{u(m)}^{2u(m)} \frac{|\ \phi_x\ (t)\ |}{t}\ (m + \tfrac{1}{2})\ t\ dt$$

(equation continued on p. 377)

$$\leqslant (m + \tfrac{1}{2}) \int_{u(m)}^{2u(m)} \omega(t)\, dt$$

$$\leqslant \pi\, \omega(2u(m)) = O(\omega(\frac{1}{m})). \qquad \text{...(2.12)}$$

Finally,

$$| I_5 | \leqslant \int_{v(m)-u(m)}^{v(m)} \frac{| \phi_x (t + u(m)) |}{t + u(m)}\, dt$$

$$= \int_{v(m)}^{v(m)+u(m)} \frac{| \phi_x(t) |}{t}\, dt.$$

By (1.10) and (1.11) we have

$$| I_5 | = O\{\sqrt{M}\, \omega(\frac{1}{m} \int_{v(m)}^{v(m)+u(m)} t^{-1/2}\, dt\}.$$

But

$$[v(m) + u(m)]^{1/2} - [v(m)]^{1/2} = [v(m)]^{1/2} \left\{(1 + \frac{u(m)}{v(m)}^{1/2} - 1\right\}.$$
$$= O\{ [u(m)]^{1/2}\}.$$

Thus

$$| I_5 | = O(\omega(\frac{1}{m})). \qquad \text{....(2.13)}$$

Hence the lemma follows from (2.3) to (2.13).

Lemma 3—Let $0 < \beta < 1$, $0 < t \leqslant \pi$ and let ρ, θ, t and β satisfy

$$\rho_\theta^{-i\theta} = 1 - \beta e^{-it} \qquad \text{...(2.14)}$$

Then, for $p = 0, 1, 2, \ldots,$

$$\text{(i)} \quad | \theta - \frac{\beta t}{1 - \beta} | \leqslant Ct^3 \qquad \text{...(2.15)}$$

$$\text{(ii)} \quad (\frac{1 - \beta}{\rho})^p \leqslant \text{exy}\, (-Apt^2) \qquad \text{...(2.16)}$$

where C and A are positive constants depending on β, and

$$\text{(iii)} \quad \left(\frac{1 - \beta}{\rho} \right)^p - \exp\left(\tfrac{1}{2} p\beta\left(\frac{t}{1-\beta}\right)^2\right) = O(pt^4). \qquad \text{...(2.17)}$$

These results are known. For example, for (i) see Miracle[7] and for (ii) and (iii) see Forbes[2].

3. Proof of the Results

Proof of Theorem 1—By (1.3), (1.4) and (1.8) we get

$$\sigma_p(x) - f(x) = \frac{1}{\pi}\int_0^\pi \frac{\phi_x(t)}{\sin \frac{1}{2}t} \sum_{k=0}^{\infty} c_k(p) \sin(k+\tfrac{1}{2})\, t\, dt + O(q^{-1/2})$$

$$= \frac{1}{\pi}\int_0^\pi \frac{\phi_x(t)}{\sin\frac{1}{2}t}\Big[\Big(\sum_{|k-q|\leqslant q^\gamma} + \sum_{|k-q|>q^\gamma}\Big)$$

$$\times\, c_k(p)\sin(k+\tfrac{1}{2})\, t\Big]\, dt + O(q^{-1/2})$$

$$= S_1 + S_2 + O(q^{-1/2}). \qquad \text{...(3.1)}$$

We first estimate S_2 :

$$|S_2| \leqslant \int_0^\pi \frac{|\phi_x(t)|}{t} \sum_{|k-q|>q^\gamma} c_k(p)\,(k+\tfrac{1}{2})\, t\, dt \qquad \text{...(3.2)}$$

$$= O(\exp(-q^\mu)) \text{ (by (1.7))}.$$

Using (1.6), S_1 can be written as follows :

$$S_1 = \frac{1}{\pi}\int_0^\pi \frac{\phi_x(t)}{\sin\frac{1}{2}t} \sum_{|k-q|<q^\gamma} g(q,k)\Big\{1 + O\Big(\frac{|k-q|+1}{q}\Big)$$

$$+ O\Big(\frac{|k-q|^3}{q^2}\Big)\Big\}\sin(k+\tfrac{1}{2})\, t\, dt$$

$$= S_3 + S_4 + S_5, \text{ say}. \qquad \text{...(3.3)}$$

We estimate S_4 and S_5 by splitting the integral into two parts and using condition (1.10) and properties of sine function.

$$|S_3| < \Big(\int_0^{1/q} + \int_{1/q}^{\pi}\Big) \frac{\phi_x(t)}{t} \sum_{|k-q|<q^\gamma} g(d,k)\, O\Big(\frac{|k-q|+1}{q}\Big)$$

$$\times \sin(k+\tfrac{1}{2})\, t\, dt$$

(*equation continued on p.* 379)

$$= O\left\{\int_0^{1/q} |\phi_x(t)| \sum_{|k-q| < q^\gamma} g(q,k)\, O\left(\frac{|k-q|+1}{q}\right) \times (|k-q|+q+\tfrac{1}{2})\, dt\right\}$$

$$+ O\left\{\int_{1/q}^{\pi} \frac{|\phi_x(t)|}{t} \sum_{|k-q| < q^\gamma} g(q,k)\left(\frac{|k-q|+1}{q}\right) dt\right\}$$

$$= O\left\{\sqrt{q}\int_0^{1/q} |\phi_x(t)|\, dt\right\} + O\left\{\frac{1}{\sqrt{q}}\int_{1/q}^{\pi} \frac{|\phi_x(t)|}{t}\, dt\right\}$$

$$= O\left(\omega\left(\frac{1}{q}\right)\right). \qquad \ldots(3.4)$$

Similarly,

$$|S_5| = O\left(\omega\left(\frac{1}{q}\right)\right). \qquad \ldots(3.5)$$

Since $m = m(p)$ is an integer valued function of p, by Lemmas 1 and 2 we have

$$\frac{1}{\pi}\int_0^{\pi} \frac{\phi_x(t)}{\sin \frac{1}{2}t} \sum_{|k-m| \leqslant m^\gamma} g(m,k) \sin(m+\tfrac{1}{2})t\, dt$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{\phi_x(t)}{\sin \frac{1}{2}t} \exp(-mt^2/4a) \sin(m+\tfrac{1}{2})t\, dt$$

$$+ O\, m^{3/2-\gamma} \exp(-am^{2\gamma-1}))$$

$$= O\left(\omega\, \frac{1}{m}\right). \qquad \ldots(3.6)$$

Now we shall estimate the difference :

$$\int_0^{\pi} \frac{\phi_x(t)}{\sin \frac{1}{2}t} \sum_{|k-q| < q^\gamma} g(q,k) \sin(k+\tfrac{1}{2})t\, dt$$

(*equation continued on p.* 380)

$$- \int_0^\pi \frac{\phi_x(t)}{\sin \frac{1}{2}t} \sum_{|k-m| \leqslant m^\gamma} g(m, k) \sin (k + \tfrac{1}{2}) t \, dt$$

$$= \int_0^\pi \frac{\phi_x(t)}{\sin \frac{1}{2} t} \sum_{m \leqslant k < m+m^\gamma} (g(q, k) - g(m, k)) \sin (k + \tfrac{1}{2}) t \, dt$$

$$+ \int_0^\pi \frac{\phi_x(t)}{\sin \frac{1}{2} t} \sum_{m-m^\gamma \leqslant k < m} (g(q,k) - g(m,k)) \sin (k+\tfrac{1}{2}) t \, dt$$

$$- \int_0^\pi \frac{\phi_x(t)}{\sin \frac{1}{2} t} \sum_{q+q^\gamma < k < m+m^\gamma} g(q, k) \sin (k+\tfrac{1}{2}) t \, dt$$

$$+ \int_0^\pi \frac{\phi_x(t)}{\mathrm{n}\frac{1}{2}t} \sum_{q-q^\gamma \leqslant k \leqslant m-m^\gamma} g(q, k) \sin (k + \tfrac{1}{2}) t \, dt$$

$$= D_1 + D_2 + D_3 + D_4. \quad \ldots(3.7)$$

First,

$$| D_3 | \leqslant \int_0^\pi \frac{| \phi_x(t)}{t} \sum_{q+q^\gamma < k \leqslant m+m^\gamma} g(q, k) \, | \sin (k + \tfrac{1}{2}) t \, | \, dt$$

$$= O \{\sqrt{q} \exp (aq^{2\gamma-1})\}. \quad \ldots(3.8)$$

Similarly

$$| D_4 | = O \{\sqrt{q} \exp (-aq^{2\gamma-1})\}. \quad \ldots(3.9)$$

For $q < m \leqslant k \leqslant m + m^\gamma$ we have

$$O \leqslant (k - m)/\sqrt{m} < (k - q)/\sqrt{q} < (k - n)/\sqrt{n}.$$

Now (cf. Ikeno[4], p.259)

$$| g(q, k) - g(m, k) | = O\left\{g(m, k)\left(\frac{(k - m)^2}{m^2} + \frac{| k-m |}{m} + \frac{1}{m} \right)\right\}. \quad \ldots(3.10)$$

Using (3.10) and the condition (1.10) on ω, we get

$$| D_1 | = O \left\{\int_0^\pi \frac{| \phi_x(t) |}{t} \sum_{m \leqslant k \leqslant m+m^\gamma} g(m, k) \left(\frac{(k - m)^2}{m^2}\right.\right.$$

(*equation contd. on p.* 381)

$$+ \frac{|k-m|}{m} + \frac{1}{m}\Big) \times |\sin(k+\tfrac{1}{2})t|\, dt\Big\}$$

$$= O\Big\{\int_0^{1/m} \frac{|\phi_x(t)|}{t} \sum_{m\leqslant k\leqslant +m^{\gamma}} g(m, k)\Big(\frac{(k-m)^2}{m^2}$$

$$+ \frac{|k-m|}{m} + \frac{1}{m}\Big)(|k+m| + m + \tfrac{1}{2})\, t\, dt\Big\}$$

$$+ O\Big\{\int_{1/m}^{\pi} \frac{|\phi_x(t)|}{t} \sum_{m\leqslant k\leqslant m+m^{\gamma}} g(m, k)\Big(\frac{(k-m)^2}{m^2}$$

$$+ \frac{|k-m|}{m} + \frac{1}{m}\Big)\, dt\Big\}$$

$$= O\Big\{\sqrt{m}\int_{1/m}^{1/m} |\phi_x(t)|\, dt\Big\} + O\Big\{\frac{1}{\sqrt{m}}\int_{1/m}^{\pi} \frac{|\phi_x(t)|}{t}\, dt\Big\}$$

$$= O(\omega(\frac{1}{m})). \qquad \ldots(3.11)$$

If $k \leqslant n < q < m$, we have

$$(k-m)/\sqrt{m} < (k-q)/\sqrt{q} < (k-n)\sqrt{n} \leqslant 0$$

and (cf. Ikeno[4], p. 260)

$$|g(q, k) - g(m, k)| = O\Big\{g(n, k)\Big(\frac{(k-n)^2}{n^2} + \frac{|k-n|}{n} + \frac{1}{n}\Big)\Big\}.$$

Using this and arguing analogous to the estimation of D_1 we get

$$|D_2| = O(\omega(\frac{1}{m})). \qquad \ldots(3.12)$$

Theorem 1 follows from (3.1) to (3.12) in view of (2.6) and (2.7).

Proof of Theorem 2—With ρ and θ defined in (2.14) we have

$$S_{\beta}^{p}(x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \frac{\phi_x(t)}{\sin\frac{1}{2}t} \sum_{k=0}^{\infty} (1+\beta)^{p+1}\binom{p-k}{p}\beta^k$$

$$\times \sin(k+\tfrac{1}{2})\, t\, dt$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{\phi_x(t)}{\sin\frac{1}{2}t}\, \mathrm{Im}\Big\{e^{it/2}\frac{(1-\beta)^{p+1}}{(1-\beta e^{it})^{p+1}}\Big\}\, dt$$

(*equation continued on p.* 282)

$$= \frac{1}{\pi} \int_0^{\pi} \frac{\phi_x(t)}{\sin \frac{1}{2}t} \left(\frac{1-\beta}{\rho} \right)^{p+1} \sin ((p+1)\theta + \tfrac{1}{2}t)\, dt$$

$$= \frac{1}{\pi} \left(\int_0^{a(p)} + \int_{a(p)}^{b(p)} + \int_{b(p)}^{\pi} \right) \frac{\phi_x(t)}{\sin \frac{1}{2}t} \left(\frac{1-\beta}{\rho} \right)^{p+1}$$

$$\times \sin ((p+1)\theta + \tfrac{1}{2}t)\, dt$$

$$= \chi_1 + \chi_2 + \chi_3. \qquad \ldots(3.13)$$

Now, using the fact that $(1 - \beta) \leqslant \rho$ as also lemma 3 (i), we have

$$|\chi_1| = O\left\{ \int_0^{a(p)} \frac{|\phi_x(t)|}{t} \left(\tfrac{1}{2}t + (p+1)\left(Ct^3 + \frac{\beta t}{1-\beta}\right)\right) dt \right\}$$

$$= O\{p \int_0^{a(p)} t^{\alpha}\, dt\}$$

$$= O(p^{-\alpha}).$$

By applying Lemma 3 (ii) to χ_3 and arguing similar to the estimation of J_3 in Lemma 2, we get

$$|\chi_3| = O\left\{ \int_{b(p)}^{\pi} \frac{|\phi_x(t)|}{t} \exp(-A(p+1)t^2)\, dt \right\}$$

$$= O\left\{ \int_{b(p)}^{\infty} \frac{1}{t} \exp(-A(p+1)t^2)\, dt \right\}$$

$$= O\left\{ \int_{A(p+1)[b(p)]^2}^{\infty} \frac{1}{u} e^{-u}\, du \right\}$$

$$= O\{p^{2\delta-1} \exp(-A p + 1)[b(p)]^2)\}$$

$$= O(p^{-\alpha}).$$

χ_2 can be estimated exactly like the 'η_2' in the context of Taylor means treated by Chui and Holland[1] (pp. 35-37). We omit the details.

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