FLOW IN A HELICAL PIPE

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Using an orthogonal coordinate system along a generic spatial curve, Germano¹, the problem of fully developed viscous flow in a helical pipe is studied. Assuming that the curvature κ and torsion τ are small compared to the radius a of the pipe, i.e., κa ≪ ε ≪ 1, τa ≪ ε λ ≪ 1, such that λ = O (1), the flow field is sought as a regular perturbation scheme of the main Poiseuille flow, in powers of ε. Analytical results obtained up to O (ε²) show that the effect of torsion on helical pipe flow is a second order one, while the effect of curvature is a first order one. It is observed that the induced second order flow field due to torsion is linearly dependent on the cotangent of the helical angle of the central axis and the corresponding velocity profiles deviate symmetrically with respect to both the conjugate diameters of the circular section of the pipe. However, the flow rate remains independent of torsion up to the order considered. In the limiting case of λ = 0, the solutions of toroidal pipe flow are recovered.

1. INTRODUCTION

Curved configurations of circular tubes—such as toroidal coils, helical coils and spiral coils—are increasingly used in industrial operations involving heat exchangers, chemical reactors, rocket engines etc. Experimental investigations have shown that the flow patterns in curved tubes are significantly different over straight tubes. Dean² was the first person to study theoretically the flow in a curved pipe using concentric toroidal coordinate system. A secondary flow which divides itself along the diameter of the tube to constitute two sets of distinct recirculating vortices was established. Superposing the secondary flow on the motion along the channel, he showed that the resulting flow of a fluid element corresponds to a skewed helical motion. These theoretical investigations of Dean² were in complete agreement with dye-injection experiments of Eustice³⁴. However, the effect of curvature was not seen in the flow rate, up to O (ε) considered, the parameter ε characterising the curvature ratio of the pipe. Subsequently Dean⁵ generated higher order terms approximately, to see the said effect. Extending
Dean's work, Topakoglu obtained exact solutions for the flow field up to \( O (\varepsilon^2) \), and significantly brought about a correction to Dean's flow rate. Numerous authors later utilised Dean's coordinates to investigate the various aspects of fluid flow in toroidal pipes. But not much of a literature is available for flow in helical pipes which involve both curvature and torsion.

Wang was the first person to study the effect of curvature and torsion on the flow in a helical pipe of circular cross-section. Introducing a non-orthogonal helical coordinate system and writing the continuity and Navier-Stokes equations for an incompressible viscous fluid in tensorial (Contravariant) form, Wang developed a perturbative analysis for the flow field in terms of the parameter \( \varepsilon \), characterising the curvature of the central helix. His analysis is valid under the assumption that both curvature \( \varepsilon \) and torsion \( \tau \) are small (i.e.)

\[
\varepsilon \alpha = \varepsilon \ll 1, \quad \tau \alpha = \varepsilon \lambda \ll 1
\]

such that the ratio \( \lambda \) of torsion to curvature is of \( O (1) \). The volumetric flow rate obtained by Wang shows that torsion does not affect the flow rate, to the \( O (\varepsilon^2) \) considered, and hence deservedly agrees with that of Topakoglu. However, his secondary flow field reported up to \( O (\varepsilon) \) was not correct as he had failed to correlate his analysis with physical covariant description. Consequently his observation of secondary flow (for non-zero torsion) reflecting asymmetrical recirculating cells is misleading. This criticism was brought out by Germano. Suitably modifying Wang's coordinate system, Germano introduced an orthogonal curvilinear coordinate system along generic spatial curve spanning the helical pipe. Deriving the governing equations of fluid flow with respect to the new frame of reference, he developed a similar analysis to that of Wang. Equations determining the first order perturbed velocity field of the main Poiseuille flow, were seen to be identical to those of Dean. Hence without solving, it was possible for Germano to disprove Wang in a simple way that torsion does not cause first order effect on motion.

It is therefore reasonable to ask at what stage and how torsion can influence the flow. The analysis of Germano is extended up to \( O (\varepsilon^3) \) in the present paper, to provide an answer to this query.

2. Orthogonal Coordinate System and Equations

The steady, laminar flow of an incompressible viscous fluid in a helically coiled pipe, at moderately low Reynolds number is considered. The fluid transportation is caused due to pressure drop along the pipe. The central axis of the pipe, with reference to an arbitrarily chosen triad OXYZ, is described by

\[
\mathbf{R}(s') = c \cos \left( \frac{s'}{(b^2 + c^2)^{1/2}} \right) \hat{i} + c \sin \left( \frac{s'}{(b^2 + c^2)^{1/2}} \right) \hat{j} + \frac{bs'}{(b^2 + c^2)^{1/2}} \hat{k}.
\]

...(2.1)
$s'$ measures the arc length along the curve, $c$ is the radius of the circular cylinder on which the helix is coiled and $2\pi b$ is the pitch. Use of Serret-Frenet formulae gives the curvature $\kappa$ and torsion $\tau$ of the central helix (2.1) as

$$
\kappa = c[(b^2 + c^2), \quad \tau = b[(b^2 + c^2). \quad \ldots(2.2)
$$

Let $\hat{T}$, $\hat{N}$, $\hat{B}$ denote the unit tangent, normal and binormal at a point $P_1$ of the central axis. The $\hat{N} - \hat{B}$ plane cuts the helical pipe in a circular section of radius $a$. The position vector of any point $P$ in this section, Wang\textsuperscript{7} is given by

$$
\vec{X}(s') = R(s') + r' \cos \theta \hat{N}(s') + r' \sin \theta \hat{B}(s') \quad \ldots(2.3)
$$

where $r'$ measures the radial distance $P_1P$ and $\theta$ the inclination of $P_1P$ with $\hat{N}$ at $P_1$. Computing the metric, one can see that the coordinate system $(s', r', \theta)$ is non-orthogonal. Making use of the fact that the origin of angle $\theta$ in the plane normal to the axis is arbitrary, Germano\textsuperscript{1} constructed an orthogonal coordinate system $(s', r', \theta + \phi (s') + \frac{1}{2}\pi)$, Fig. 1, where $\theta$ measures now the inclination of the radius

![Fig. 1. Germano’s coordinate system.](image-url)
vector $P_1P$ to an arbitrary direction in the normal section, the direction being characterised as inclined to the principal normal $\hat{N}$ at an angle $\frac{1}{2}\pi + \phi (s')$, with $\phi$ being defined by

$$\phi (s') = - \int_0^{s'} \tau (s') \, ds'.$$  \hspace{1cm} (2.4)

$\vec{q} = (u', v', w')$ denotes the fluid velocity at $P$. In terms of dimensionless variables,

$$s = s'|a, r = r'|a, u = u'|U, v = v'|U, w = w'|U, p = p'|pU^2$$  \hspace{1cm} (2.5)

the equations of momentum and continuity in Germano's system are given as

$$Du + \epsilon u \left[ \nu \sin (\theta + \phi) + w \cos (\theta + \phi) \right] = - \frac{\omega}{\Re} \frac{\partial p}{\partial s}$$

$$+ \frac{1}{\Re} \left[ \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \left( \frac{\partial u}{\partial r} + \epsilon \omega \sin (\theta + \phi) - \omega \frac{\partial v}{\partial s} \right) \right.$$  \hspace{1cm} (2.6a)

$$+ \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \epsilon \omega \cos (\theta + \phi) - \omega \frac{\partial w}{\partial s} \right) \right]$$

$$Dv - \frac{w^2}{r} - \epsilon \omega u^2 \sin (\theta + \phi) = - \frac{\partial p}{\partial r}$$

$$- \frac{1}{\Re} \left[ \left( \frac{1}{r} \frac{\partial}{\partial \theta} + \epsilon \omega \cos (\theta + \phi) \right) \left( \frac{\partial w}{\partial r} + \frac{w}{r} - \frac{1}{r} \frac{\partial v}{\partial \theta} \right) \right.$$  \hspace{1cm} (2.6b)

$$- \omega \frac{\partial}{\partial s} \left( \omega \frac{\partial v}{\partial s} - \frac{\partial u}{\partial r} - \epsilon \omega u \sin (\theta + \phi) \right) \right]$$.  \hspace{1cm} (2.6b)

$$Dw + \frac{v w}{r} - \epsilon \omega u^2 \cos (\theta + \phi) = - \frac{1}{\Re} \frac{\partial p}{\partial \theta}$$

$$+ \frac{1}{\Re} \left[ \left( \frac{\partial}{\partial r} + \epsilon \omega \sin (\theta + \phi) \right) \left( \frac{\partial w}{\partial r} + \frac{w}{r} - \frac{1}{r} \frac{\partial v}{\partial \theta} \right) \right.$$  \hspace{1cm} (2.6c)

$$- \omega \frac{\partial}{\partial s} \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \epsilon \omega \cos (\theta + \phi) - \omega \frac{\partial w}{\partial s} \right) \right]$$

$$+ \omega \frac{\partial u}{\partial s} + \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial w}{\partial \theta} + \frac{v}{r} + \epsilon \omega \left[ \nu \sin (\theta + \phi) + w \cos (\theta + \phi) \right] = 0$$  \hspace{1cm} (2.6d)

where

$$D = \omega u \frac{\partial}{\partial s} + v \frac{\partial}{\partial r} + \frac{w}{r} \frac{\partial}{\partial \theta}$$

$$\omega = \frac{1}{[1 + \epsilon r \sin (\theta + \phi)]}, \Re = \frac{Ua}{\nu}.$$  \hspace{1cm} (2.7)

The reference velocity $U$ is the central velocity of the main Poiseuille flow.
The boundary conditions are the usual no-slip conditions on the body, viz.,
\[ \bar{q} = (0,0,0) \text{ on } r = 1. \] ... (2.8)

3. **Solutions—First Approximation**

It is assumed that both curvature and torsion are small, but their ratio is appreciable i.e.
\[ \epsilon \alpha = \epsilon \ll 1, \tau a = \epsilon \lambda \ll 1, \lambda = O(1). \] ... (3.1)

Taking the Reynolds number \( R_e \) to be of \( O(1) \) and neglecting the end effects, the description of the flow through the helical pipe can be obtained as a perturbation of the main Poiseuille flow. Hence one can look for solutions describing the fully developed helical flow in the form
\[ u = u_0 (r) + \epsilon u_1 (r, \alpha) + \epsilon^2 u_2 (r, \alpha) + \ldots \] ... (3.2a)
\[ v = \epsilon v_1 (r, \alpha) + \epsilon^2 v_2 (r, \alpha) + \ldots \] ... (3.2b)
\[ w = \epsilon w_1 (r, \alpha) + \epsilon^2 w_2 (r, \alpha) + \ldots \] ... (3.2c)
\[ p = p_0 (s) + \epsilon p_1 (r, \alpha) + \epsilon^2 p_2 (r, \alpha) + \ldots \] ... (3.2d)

where
\[ \theta + \phi = \alpha. \] ... (3.3)

The primary flow obtained as the zeroth order of the equations (2.6) is the well known Poiseuille flow, viz.
\[ u_0 (r) = 1 - r^2 \] ... (3.4a)
\[ p_0 (s) = - \frac{4}{R_e} s. \] ... (3.4b)

The equations determining \( u_1, v_1, w_1 \) and \( p_1 \) may be seen to be identical to those of Dean² and hence the corresponding solutions are
\[ u_1 = U_1 (r) \sin \alpha, \] ... (3.5)
\[ v_1 = V_1 (r) \sin \alpha, \] ... (3.6)
\[ w_1 = W_1 (r) \cos \alpha, \] ... (3.7a)
\[ p_1 = P_1 (r) \sin \alpha, \] ... (3.8)

where
\[ U_1 (r) = r (1 - r^2) \left[ - \frac{3}{4} + \frac{R_e^2}{11520} (19 - 21r^3 + 9r^4 - r^4) \right] \] ... (3.7a)
\[ V_1 (r) = \frac{R_e}{288} (1 - r^2)^2 (4 - r^2) \quad \ldots (3.7b) \]

\[ W_1 (r) = \frac{R_e}{288} \cdot (1 - r^2) (4 - 23r^2 + 7r^4) \quad \ldots (3.7c) \]

and

\[ P_1 (r) = \frac{1}{12} r (9 - 6r^2 + 2r^4). \quad \ldots (3.7d) \]

It may be noted that the solutions obtained so far are independent of \( \lambda \), unlike that of Wang\(^2\).

Hence to bring out the effect of torsion on the flow, one has to obtain higher order terms, hitherto not studied by the earlier workers.

4. Solutions—Second Approximation

The second order terms \((u_z, v_z, w_z, p_z)\) of (3.2) are the solutions of the following set of equations:

\[ v_2 \frac{d u_0}{d r} - \lambda u_0 U_1 \cos \alpha + V_1 \frac{d U_1}{d r} \sin^2 \alpha + \frac{U_1 W_1}{r} \cos^2 \alpha \]

\[ + u_0 V_1 \sin^2 \alpha + u_0 W_1 \cos^2 \alpha \]

\[ = \lambda P_1 \cos \alpha - r^2 \frac{d p_0}{d s} \sin^2 \alpha + \frac{1}{R_e} \left[ \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \frac{\partial u_0}{\partial r} + \frac{1}{r} \frac{\partial u_1}{\partial \alpha} \right] \]

\[ + \frac{U_1}{r} \cos 2\alpha - u_0 \cos 2\alpha - \frac{\lambda W_1}{r} \cos \alpha \quad \ldots (4.1a) \]

\[ r u_0^2 \sin^2 \alpha - 2u_0 U_1 \sin^2 \alpha - \frac{W_1^2}{r} \cos^2 \alpha - \lambda u_0 V_1 \cos \alpha \]

\[ + V_1 \frac{d V_1}{d r} \sin^2 \alpha + \frac{V_1 W_1}{r} \cos^2 \alpha \]

\[ = - \frac{\partial p_2}{\partial r} - \frac{1}{R_e} \left[ \frac{1}{r} \frac{\partial}{\partial z} \left( \frac{w_2}{r} + \frac{\partial w_2}{\partial z} - \frac{1}{r} \frac{\partial v_2}{\partial \alpha} \right) \right] \]

\[ + \left( \frac{W_1}{r} + \frac{d W_1}{d r} - \frac{V_1}{r} \right) \cos^2 \alpha - \lambda u_0 \cos \alpha - \lambda \frac{d U_1}{d r} \cos \alpha \] \ldots (4.1b)

\[ V_1 \frac{d W_1}{d r} \sin \alpha \cos \alpha + \frac{V_1 W_1}{r} \sin \alpha \cos \alpha - \frac{W_1^2}{r} \sin \alpha \cos \alpha \]

(equation continued on p. 81)
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\[ + \lambda u_0 W_1 \sin \alpha - 2u_0 U_1 \sin \alpha \cos \alpha + ru_0^2 \sin \alpha \cos \alpha \]

\[ = \frac{1}{r} \frac{\partial p_2}{\partial r} + \frac{1}{r} \frac{\partial w_2}{\partial r} \left( \frac{w_2}{r} + \frac{\partial w_2}{\partial r} \right) \]

\[ + \frac{W_1}{r} \sin \alpha \cos \alpha + \frac{dW_1}{dr} \sin \alpha \cos \alpha - \frac{V_1}{r} \sin \alpha \cos \alpha \]

\[ - \lambda u_0 \sin \alpha - \frac{\lambda U_1}{r} \sin \alpha \]

\[ \frac{\partial v_2}{\partial r} + \frac{v_2}{r} + \frac{1}{r} \frac{\partial w_2}{\partial r} - \lambda U_1 \cos \alpha + V_1 \sin^2 \alpha + W_1 \cos^2 \alpha = 0. \]

...(4.1c)

...(4.1d)

These equations suggest one to look for solutions in the form

\[ u_2 (r, \alpha) = U_{20} (r) + U_{21} (r) \cos \alpha + U_{22} (r) \cos 2 \alpha \]

...(4.2a)

\[ v_2 (r, \alpha) = V_{20} (r) + V_{21} (r) \cos \alpha + V_{22} (r) \cos 2 \alpha \]

...(4.2b)

\[ w_2 (r, \alpha) = W_{20} (r) + W_{21} (r) \sin \alpha + W_{22} (r) \sin 2 \alpha \]

...(4.2c)

\[ p_2 (r, \alpha) = P_{20} (r) + P_{21} (r) \cos \alpha + P_{22} (r) \cos 2 \alpha \]

...(4.2d)

subject to the conditions that they remain finite as \( r \to 0 \) and vanish on \( r = 1 \).

The respective solutions \( U_{2j}, V_{2j}, W_{2j} \) and \( P_{2j} \) \((j = 0, 1, 2)\) are obtainable following a straightforward sequential procedure, though involving voluminous algebra.

The consolidated results for the velocity and pressure fields up to \( O (\varepsilon^2) \) are given below.

\[ u = (1 - r^2) \left[ 1 + \varepsilon r \left\{ - \frac{3}{4} + \frac{R_e^6}{11520} (19 - 21r^2 + 9r^4 - r^6) \right\} \right] \sin \alpha \]

\[ + \varepsilon^2 \left\{ - \frac{1}{32} (3 - 11r^2) - \frac{R_e^4}{230400} (148 + 43r^2 - 132r^4 \]

\[ + 68r^6 - 7r^8) - \frac{R_e^4}{3715891200} (4119 - 17161r^2 + 29179r^4 \]

\[ - 26261r^8 + 13569r^8 - 4015r^{10} + 605r^{12} - 35r^{14}) \right\} + \lambda r \left( \frac{R_e^2}{576} \right)^{29} \]

\[ + 5r^2 - 3r^4 + \frac{R_e^3}{29030400} (2969 - 4381r^2 + 3249r^{14} \]

\[ - 1301r^6 + 274r^8 - 20r^{10}) \right\} \cos \alpha + \left( \frac{5}{16} \right) + \frac{R_e^2}{276480} \]

(equation continued on p. 82)
\[
\begin{align*}
(463 - 613r^2 + 296r^4 - 40r^6) - \frac{R_e^4}{117050572800} (145690 \\
- 240206r^2 + 174649r^4 - 70547r^6 + 19123r^8 - 2801r^{10} \\
+ 160r^{12}) \cos 2\alpha & + O(\epsilon^3) \quad \ldots \quad (4.5a)
\end{align*}
\]

\[
\begin{align*}
v &= (1 - r^2)^2 \left[ \epsilon \frac{R_e}{288} (4 - r^2) \sin \alpha + \epsilon^2 \left\{ - \frac{R_e}{576} r (4 - r^2) \\
+ \lambda \left( \frac{1}{6} + \frac{R_e^2}{69120} (13 - 15r^2 + 7r^4 - r^6) \right) \cos \alpha \\
+ r \left( \frac{R_e}{240} (3 - r^2) - \frac{R_e^3}{232243200} (4979 - 2792r^2 + 777r^4 \\
- 134r^6 + 5r^8) \right) \cos 2\alpha \right\} \right] + O(\epsilon^3) \quad \ldots \quad (4.5b)
\end{align*}
\]

\[
\begin{align*}
w &= (1 - r^2) \left[ \epsilon \frac{R_e}{288} (4 - 23r^2 + 7r^4) \cos \alpha + \epsilon^2 \left\{ \lambda \left( - \frac{1}{12} \\
(2 - r^2) - \frac{R_e^2}{69120} (13 - 224r^2 + 266r^4 - 124r^6 + 17r^8) \right) \sin \alpha \\
+ r \left( \frac{-R_e}{960} (12 - 59r^2 + 21r^4) + \frac{R_e^3}{232243200} (4979 - 20521r^2 \\
+ 13499r^4 - 4421r^6 + 829r^8 - 35r^{10}) \right) \sin 2\alpha \right\} \right] + O(\epsilon^3) \quad \ldots \quad (4.5c)
\end{align*}
\]

\[
\begin{align*}
p &= - \frac{4}{R_e} s + \epsilon \frac{r}{12} (9 - 6r^2 + 2r^4) \sin \alpha + \epsilon^2 \left\{ - \frac{r^2}{144} (81 \\
- 81r^2 + 28r^4) + \frac{R_e^2}{1382400} r^2 (1140 - 1095r^2 + 200r^4 \\
+ 225r^6 - 108r^8 + 15r^{10}; \\
+ \lambda r \left( - \frac{1}{6R_e} (1 - 3r^2) + \frac{R_e}{17280} (101 - 120r^2 + 90r^4 \\
- 30r^6 + 3r^8) \right) \cos \alpha + r^2 \left( \frac{1}{120} (54 - 55r^2 + 20r^4) \\
- \frac{R_e^2}{3870720} (3597 - 8344r^2 + 9240r^4 - 5040r^6 + 1288r^8 \\
- 120r^{10}) \right) \cos 2\alpha \right\} \right] + O(\epsilon^3). \quad \ldots \quad (4.5d)
\end{align*}
\]
5. Flow Rate

The volume rate of discharge of the fluid through the circular cross section of the pipe is given in terms of dimensionless variables as

\[
\frac{\dot{q}}{Ua^2} = \int_0^1 \int_0^{2\pi} ur \, dx \, dr
\]

(5.1)

Computing the integral (5.1), one obtains the flow rate \[Q = \dot{q}/2\pi \, Ua^2\] as

\[
\frac{Q}{Q_*} = 1 - \frac{\epsilon^2}{48} \left[ -\frac{1541}{67200} \left( \frac{Re}{6} \right)^4 + \frac{11}{10} \left( \frac{Re}{6} \right)^2 - 1 \right] + O(\epsilon^3)
\]

(5.2)

where \(Q_*\) is the respective flow rate in a straight tube.

6. Discussion

The flow rate (5.2) obtained comparatively with lesser effort using Germanos'\(^1\) Orthogonal Coordinate system, agrees identically with Wang's\(^7\) result. The terms involving \(\lambda\) in the expression (4.5a) for \(u\) are periodic in \(\alpha\), hence the reason for the flow rate (5.1) to be independent of torsion. Naturally the expression (5.2) agrees with the result of Topakoglu\(^6\) for a toroidal pipe. Owing to enhanced mixing and momentum transfer due to secondary flows, it is common belief that the total frictional loss of energy near the wall increases and consequently the fluid experiences more resistance in passing through the circular/helical pipe. Looking at this angle, in the limit of small \(\epsilon\), it can be seen from the analysis that when \(Re > 5.67\), the flux in the helical pipe is less than that of a straight pipe. However for the flow to remain laminar, a rough estimation yields an upper bound for \(Re\) as \(Re \approx 40/\sqrt{\epsilon}\). The corresponding Dean number \(\left(2\pi R_e^2\right)\) is approximately 3200.

The computed flow field (4.5), enables one to verify analytically that the effect of torsion on a helical pipe flow is a second order one, while the effect of curvature is a first order one. It can be observed that the induced second order flow is linearly dependent on the cotangent of the helical angle of the central generic spatial curve. As the terms involving \(\lambda\) are all periodic in \(\alpha\), the presence of torsion causes a symmetric deviation of the velocity profiles, not only with respect to the diameter of full circle circle symmetry (a diameter of the circular section containing the principal normal) but also with respect to the conjugate diameter. However the full profiles of the velocities remain symmetric only with respect to the diameter of the full circle symmetry. It can be seen that the flow field (4.5) reported in this paper is finer and exact up to the order considered compared to the approximated results of Murata et al.\(^8\). By putting \(\lambda = 0\) one can recover the flow field of Topakoglu\(^6\), for toroidal pipe flow.

If terms of \(O(\epsilon^3)\) are taken into consideration, in view of non-zero torsion it is not possible to express the secondary velocities in terms of a stream function and con-
sequently project the streamline flow pattern. However, confining to $O(\varepsilon)$, one can express the secondary velocities in the $(r, \theta)$ plane in terms of stream function $\psi$ as

$$\frac{\psi}{R_e} = -\frac{1}{288} \left(4r - 9r^3 + 6r^5 - r^7\right) \cos \alpha$$

...(6.1)

where

$$v_1 = \frac{1}{r} \frac{\partial \psi}{\partial \alpha}, \ w_1 = -\frac{\partial \psi}{\partial r}.$$  

...(6.2)

The expression (6.1) is independent of $\lambda$, thereby showing (as observed earlier) that torsion has no first order effect on the flow. This disproves Wang$^7$, who erroneously calculated the secondary flows and predicted asymmetric recirculating cells which tend to coalesce with decreasing $R_e$, for non-zero torsion. This was due to the fact that Wang$^7$ failed to correlate the contravariant components of the velocity vector to a physical covariant description, as remarked by Germano$^1$. Fig. 2 projects the streamlines

![Diagram of streamlines](image)

Fig. 2. Secondary streamlines. $\psi/R_e$=constant.
\[ \psi \pm 0.0037, \pm 0.0035, \pm 0.0030, \pm 0.0020, \pm 0.0010, \pm 0.0005, \pm 0.000137 \] in the \((r, \theta)\) plane. It can be seen easily from the diagram that the secondary flow field divides itself along the diametral axis of the section of the pipe into two mirror images of recirculating cells. Obviously these streamlines are the same as that of Wang\(^7\), for the case \(\lambda = 0\). The vortex centre (point of no secondary flow) observed at \(r = 0.43\), \(\alpha = \pi/2\), though well away from the origin, is still closer to the diameter of the full circle symmetry than the upper wall. As expected, this observation is in agreement with Dean\(^8\) and Austin and Seader\(^9\).

**References**