AN ALGEBRAICALLY SPECIAL BIANCHI TYPE VI$_h$ COSMOLOGICAL
MODEL IN GENERAL RELATIVITY

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In this paper a Bianchi type VI$_h$ space-time filled with a viscous fluid is con-
didered which is of Petrov type II. Various physical properties have also been
discussed.

1. Introduction

Bianchi type VI$_h$ universes representing perfect fluid distribution have been
studied by Collins$^2$ and Wainwright and Anderson$^6$. It is well known that such
universes do not go over to FRW universes asymptotically but they can be made as
close to such models as one wants them to be in a finite interval of time, the latter cor-
responding either to the very early stage of the universe or a late or suitably inter-
mediate stage in the evolution of the actual universe. In the early stage of the universe
mater behaved like a viscous fluid during the period of neutrino-decoupling. It is also
expected that during the big bang explosion copious amount of gravitational radiation
was produced$^1$.

In this paper, we have derived a Bianchi type VI$_h$ universe which is filled with a
viscous fluid and which is of Petrov type II, the latter condition ensuring that the
universe contains gravitational radiation. The model so constructed gives rise to two
different cases, in each of which the expanding and the contracting phases of the
universe are described separately by two different line-elements joining smoothly at the
point of no expansion. In each case the universe expands from an initial singularity
till the expansion stops and thereafter contracts and collapses into a final singularity.
The model in some special situations can be regarded as close to FRW universe
during the early stages of its evolution. The relative behaviour of electric and magnetic
parts of the free gravitational field near both the singularities have been studied and
it is found that in one case the gravitational field is of type $N$ near the final singularity.

2. Derivation of the Line-element

We have taken the line-element which describes Bianchi type VI$_h$ space-times in
the form

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)e^{2x}dy^2 + C^2(t)e^{2z}dz^2 \quad \text{...(2.1)}$$
where $h = \pm 1$.  

We assume that the universe is filled with a viscous fluid, the energy-momentum tensor $T_{ij}$ of which is given by  

$$T_{ij} = (\varepsilon + \dot{p}) v_i v_j + \mu g_{ij} - 2\eta \sigma_{ij}$$  

with  

$$\dot{p} = p - \zeta \dot{\theta};$$  

$p$, $\varepsilon$, $\eta$, $\zeta$, $\theta$, $\sigma_{ij}$ and $v_i$ being respectively the pressure, density, coefficients of shear and bulk viscosities, expansion scalar, shear tensor and the unit flow vector of the fluid assumed to be orthogonal to the hypersurfaces of homogeneity. 

The field equations  

$$R^i_i - \frac{1}{2} R g^i_i + \wedge g^i_i = -8\pi T^i_i$$  

lead to  

$$-8\pi \rho - \Lambda = \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{BC}}{BC} - \frac{2}{3} \eta \left( \frac{2}{A} - \frac{B}{B} - \frac{C}{C} \right) - \frac{h}{A^2}$$  

$$= \frac{\ddot{C}}{C} + \frac{\dot{A}}{A} + \frac{CA}{CA} - \frac{2}{3} \eta \left( -\frac{A}{A} + 2\frac{B}{B} - \frac{C}{C} \right) - \frac{h^2}{A^2}$$  

$$= \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{AB}}{AB} - \frac{2}{3} \eta \left( -\frac{A}{A} - \frac{B}{B} + 2\frac{C}{C} \right) - \frac{1}{A^2}$$  

$$(2.2)$$  

$$8\pi \epsilon - \Lambda = \frac{\dot{AB}}{AB} + \frac{\dot{BC}}{BC} + \frac{\dot{CA}}{CA} - \frac{h^2 + h + 1}{A^2}$$  

$$= \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}$$  

$$(2.3)$$  

an overhead dot standing for ordinary differentiation with respect to $t$. 

From (2.4), we get  

$$A^2 = K^2 \mu \nu^{1-h/1+h}$$  

$$(2.5)$$  

where $\mu = BC$, $\nu = B/C$ and $K$ is a constant, Equations (2.2) and (2.5) lead to the single equation  

$$\frac{d}{dt} \left( \frac{\dot{v}}{v} \right) + \frac{\dot{v}}{2v} \left( 3 \frac{\mu}{\mu} + \frac{1 - h}{1 + h} \frac{\nu}{\nu} \right) + \frac{h^2 - 1}{A^2} = -16\pi \eta \frac{\dot{v}}{v}$$  

$$(2.6)$$
in three unknowns $\mu$, $\nu$ and $\eta$. In order to get a determinate solution we assume the following:

(1) The metric (2.1) is of Petrov type II,

(2) $\eta$ is proportional to $|\theta|$.

Assumption (1) leads to

$$
\frac{d}{dt} \left( \frac{\nu}{\nu} \right) + \frac{\nu}{2\nu} \left( \frac{\mu}{\mu} - \frac{1 - h}{1 + h} \cdot \frac{\nu}{\nu} \right) - \frac{h^2 - 1}{A^2} = \frac{4h}{(1 + h)} \cdot \frac{1}{A} \cdot \frac{\nu}{\nu}
$$

while (2) gives

$$
\eta = M \theta \text{ for } \theta > 0
$$

and

$$
\eta = -M \theta \text{ for } \theta < 0.
$$

$M$ being a positive dimensionless constant. Equations (2.6) and (2.7) under the condition (2.8a) give the following solution:

$$
A = L_1 \cdot \frac{\{(1 + h)^2 + (1 - h)^2 \cdot e^{\alpha T} \cdot \alpha^{1/2} \cdot e^{\nu T} \}}{\{f(1 - e^{\alpha T})^{1/2} (\alpha + \beta_1 + \gamma_1)\}}
$$

$$
B = L_2 \cdot \frac{\{(1 + h)^2 + (1 - h)^2 \cdot e^{\alpha T} \cdot \alpha^{1/2} \cdot e^{\nu T} \}}{\{f(1 - e^{\alpha T})^{1/2} (\alpha + \beta_2 + \gamma_2)\}}
$$

$$
C = L_3 \cdot \frac{\{(1 + h)^2 + (1 - h)^2 \cdot e^{\alpha T} \cdot \alpha^{1/2} \cdot e^{\nu T} \}}{\{f(1 - e^{\alpha T})^{1/2} (\alpha + \beta_3 + \gamma_3)\}}
$$

in which $f = +1$ or $-1$ according as $h > -1$ or $h < -1$, $L_i$ are arbitrary constants and the constants $q$, $\alpha$, $\beta_i$ and $\gamma_i$ are given by

$$
q = -\frac{2(1 + h^2)}{(1 + h)}, \quad \alpha = -\frac{2}{(1 + 24\pi M)}
$$

$$
\beta_1 = \frac{(1 + 8\pi M)(1 + h)^2 - 2h}{(1 + h^2)(1 + 24\pi M)}, \quad \beta_2 = \frac{8\pi M(1 + 2h)(1 + h^2)(1 + 3h^2)}{(1 - h)(1 + h^2)(1 + 24\pi M)}
$$

$$
\beta_3 = \frac{8\pi M(1 + 2h)(1 - h) + (1 + h)}{(1 + h^2)(1 + 24\pi M)}, \quad \gamma_1 = \frac{(1 + 8\pi M)(1 - h)^2 + 2h}{(1 + h^2)(1 + 24\pi M)}
$$

The same set of equations under the condition (2.8b) gives

$$
A = \overline{L_1} \cdot \frac{\{(1 + h)^2 + (1 - h)^2 \cdot e^{\alpha T} \cdot \alpha^{1/2} \cdot e^{\nu T} \}}{\{f(1 - e^{\alpha T})^{1/2} (\alpha + \beta_3 + \gamma_3)\}}
$$

$$
A = \overline{L_1} \cdot \frac{\{(1 + h)^2 + (1 - h)^2 \cdot e^{\alpha T} \cdot \alpha^{1/2} \cdot e^{\nu T} \}}{\{f(1 - e^{\alpha T})^{1/2} (\alpha + \beta_3 + \gamma_3)\}}
$$
\[ B = \overline{L}_2 \cdot \frac{\{(1 + h)^2 + (1 - h)^2 \ e^{\theta T} \}^{\frac{\alpha}{2}} \cdot e^{\frac{\alpha}{2} \ \overline{\beta} \ T}}{\{f(1 - e^{\theta T})\}^{1/2(\alpha + \overline{\beta} + \gamma_2)}} \]  
\[ C = \overline{L}_3 \cdot \frac{\{(1 + h)^2 + (1 - h)^2 \ e^{\theta T} \}^{\frac{\beta}{2}} \cdot e^{\frac{\beta}{2} \ \overline{\gamma} \ T}}{\{f(1 - e^{\theta T})\}^{1/2(\alpha + \overline{\beta} + \gamma_3)}} \]

in which \( \overline{L}_l \) are arbitrary constants and the constants \( \overline{\alpha}, \overline{\beta}_l \) and \( \overline{\gamma}_l \) are given by the corresponding unbarred constants of the solution (2.9) when \( M \) is replaced by \(-M\). \( T \) in both the solutions (2.9) and (2.10), is related to the cosmic time \( t \) by \( t = \int AdT \).

The line-element (2.1) corresponding to both the solutions (2.9) and (2.10) has singularities at \( T = 0 \) and \( T = \pm \infty \). The expression for scalar of expansion \( \theta \) corresponding to solution (2.9) is given by

\[ \theta = \frac{f \ e^{-\beta T}}{L_1 (1 + h)(1 + 24\pi M)} \cdot \frac{\{f(1 - e^{\theta T})\}^{\frac{\alpha + \beta_1 + \gamma_1}{2} - 1}}{\{(1 + h)^2 + (1 - h)^2 \ e^{\theta T} \}^{\frac{\alpha}{2} + 1}} \times [(1 + h^2 - h - H) + (1 + h^2 + h + H) \ e^{\theta T}] \]
\[ \times [(1 + h^2 - h + H) + (1 + h^2 - h - H) \ e^{\theta T}] \]

where \( H = \sqrt{3} (1 + h^2 + h^4) \). Similar expression can be written for the expansion scalar \( \overline{\theta} \) corresponding to the solution (2.10). We assume that \( (1 - 24\pi M) > 0 \). From the expressions for \( \theta \) and \( \overline{\theta} \) we find that they vanish at \( T = T' \) and \( T = T'' \), where

\[ T' = \frac{1}{q} \ \log \left( \frac{H - h^2 + h - 1}{H + h^2 + h + 1} \right) \] and \( T'' = \frac{1}{q} \ \log \left( \frac{H + h^2 - h + 1}{H - h^2 - h - 1} \right) \).

When \( h > -1 \), the model of the universe is described by the line-element (2.1) corresponding to solutions (2.9) and (2.10) in the time-ranges \( 0 < T < T' \) and \( T > T' \) respectively. Similarly, for \( h < -1 \), it is described by the solutions (2.9) and (2.10) respectively in the time-spans \( 0 < T < T'' \) and \( T > T'' \). The line-elements corresponding to both the solutions (2.9) and (2.10) join smoothly at the common boundary \( T = T' \) or \( T'' \) in the two cases mentioned above. This requires\(^4\) that across the surface of discontinuity \( x^4 \equiv T = T' \) or \( T'' \), the following are continuous: \( A, B, C, dA/dT, dB/dT, dC/dT \) and \( T^4 \). These conditions are found to be satisfied provided

\[ \overline{L}_t = L_t \cdot \frac{\{(1 + h)^2 + (1 - h)^2 \ e^{\theta T} \}^{\frac{\alpha}{2} - \alpha} \cdot e^{\frac{\alpha}{2} \ \overline{\beta} \ \overline{\gamma}_i \ T}}{\{f(1 - e^{\theta T})\}^{1/2(\alpha + \overline{\beta} + \overline{\gamma} \ i - \overline{\beta} \ i - \overline{\gamma} \ i)}} \]

where \( \tau \) stands for \( T' \) or \( T'' \) and \( i = 1, 2, 3 \). For \( h > -1 \), the model evolves with a big bang at \( T = 0 \) and expands till \( T = T' \). Thereafter it contracts till it collapses at \( T = + \infty \). Similarly for \( h < -1 \), the model evolves with a big bang at \( T = 0 \),
expands till \( T = T' \) and after that contracts to collapse at \( T = + \infty \). In terms of the cosmic time \( t \), the initial singularity occurs at \( t = 0 \) and the collapse occurs after a finite lapse of time in each case considered above. The types of the initial singularity are restricted to those of cigar and infinite pan cake of first and second kind\(^2\), while those of the final singularity are restricted to point, barrel and cigar depending on the values of \( h \) and \( M \).

The anisotropy \( a/\theta \) is finite at the start. It increases for \( h \geq 0 \) and \( h < -1 \), but for \( -1 < h < 0 \), it decreases and attains its minimum at \( T = \bar{T} \), where

\[
\bar{T} = \frac{1}{q} \log \left[ \frac{(1 + h)^2 (h^2 - h + 1)}{(1 - h)^2 (h^2 + h + 1)} \right].
\]

By a suitable choice of \( h \) the anisotropy can be made to be small in the interval \((\bar{T} - \delta, \bar{T} + \delta)\) for a given \( \delta > 0 \) so that the model is approximately FRW in this interval. We also find that the electric part of the Weyl tensor is dominant over its magnetic part except at the collapse when \( h > -1 \). In the latter case the model is asymptotically Petrov type \( N \). The condition \( \epsilon > 0 \) requires necessarily that \( h < 0 \) and

\[
M < \left[ \frac{1}{24\pi} \cdot \left\{ \frac{(1 - h)}{(1 + h + h^2)^{1/2}} - 1 \right\} \right] \text{ when } \Lambda = 0.
\]

All the above discussions hold good for the solution corresponding to perfect fluid when \( M = 0 \). In this case the reality condition \(- \epsilon < p \leq \epsilon \) is satisfied for \(-1 < h < 0 \) together with a suitable choice of \( \Lambda \).

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References