SPECTRAL IN Variant OF THE Zeta Function
OF THE LAPLACIAN ON $S^{4r-1}$

N. Sthanumoorthy

Ramanujan Institute for Advanced Study in Mathematics
University of Madras, Madras 600 005

(Dedicated to Professor T. S. Bhanu Murthy on his 60th Birthday)

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The aim of this paper is to compute a spectral invariant of the zeta function
$\zeta(A, s)$ at $s = 0$ of the Laplace Beltrami operation $\Delta$ acting on forms of
degree 2 on $(4r - 1)$ dimensional sphere $S^{4r-1}$.

1. Introduction

Let $A$ be an elliptic pseudo-differential operator of positive order $m$ on a compact $n$-dimensional manifold $X$. If $A$ is self adjoint and positive, its eigen values are $\lambda > 0$. We define its Zeta function by

$$\zeta (A, s) = \text{Trace } A^{-s} = \sum_{\lambda > 0} \lambda^{-s} \quad (\text{Atiyah et al.}).$$

Here each eigen value is repeated as many times as its multiplicity. This series converges for $\text{Re} \, (s) > \frac{n}{m}$ (Atiyah et al.) and hence gives a holomorphic function of the complex variable $s$ in this half plane. Also $\zeta (A, s)$ can be analytically continued to the whole $s$-plane as a meromorphic function with simple poles. On the other hand, for any elliptic self adjoint pseudo differential operator $B$ of positive order $m$ on a compact manifold $X$, we define its eta function as

$$\eta (B, s) = \sum_{\lambda \neq 0} \text{Sign } \lambda \mid \lambda \mid^{-s}.$$

Here also $\lambda$ runs over the eigen values of $B$ and each eigen value repeats as many times as its multiplicity. But here as the operator $B$ is not positive, its eigen values may be positive or negative and hence each eigen value is taken with its sign. When the operator $B$ is also positive, we have

$$\zeta (B, s) = \eta (B, s).$$

The real valued invariants of the metric satisfying the condition that it is a continuous function of the metric can be obtained by evaluating $\zeta$ at some point $s$ where it is known to be finite. For positive self adjoint elliptic operators, the zeta function defined above have finite values at $s = 0$ by the results in Seeley. Atiyah et al., gave two methods to prove the finiteness of $\eta$ at $s = 0$ for the special operator $B^*$. Using
Cobordism theory one can prove the finiteness of $\eta(0)$ and by using invariant theory, $\eta(0)$ can be computed explicitly for $B_+$. But in this paper we show a general method of analytic continuation to compute $\zeta(\Delta, s)$ at $s = 0$ for the Laplacian $\Delta$ and compute it for $\Delta$ acting on 2-forms on $S^{4r-1}$. The importance of this paper lies in the analytic continuation that we have used for the function of the form $\sum_{n=1}^{\infty} g(n) \{ f(n) \}^{-s}$, where $f(n)$ and $g(n)$ are two polynomials of finite degrees. The method of analytic continuation that we have used in this paper can be used for all similar series.

2. Eigenvalues of $\Delta$ on 2-forms on $S^{4r-1}$

When $G$ is a compact connected Lie group and $K$ is a closed subgroup, we consider the quotient space $M = G/K$. The Laplace Beltrami operator or Laplacian is $\Delta = d\delta + \delta d$, where $d$ is exterior differentiation and $\delta$ is operator adjoint to $d$. $\Delta$ is a self-adjoint and elliptic differential operator on $C^\infty(\Lambda^p M)$ for each $p$. Here $C^\infty(\Lambda^p M)$ is the vector space of smooth sections of the vector bundle $\Lambda^p M$, the $p$th exterior power of the complexified cotangent bundle of $M$. The set of eigenvalues of $\Delta$ on $C^\infty(\Lambda^p M)$ is a discrete set of real numbers.

When $G$ is a compact semisimple Lie group and $B$ is the killing form of the Lie algebra $G$ of $G$, the Casimir element is $C = \sum_{1 \leq i, j \leq N} c_{ij} X_i \cdot X_j$ where $\{X_1, \ldots, X_N\}$ is a basis of $\mathfrak{g}$ and $c_{ij} = (B(X_i, X_j))^{-1}$. When $(G, K)$ is a compact symmetric pair with a compact connected semisimple Lie group $G$, we have the Cartan decomposition

$$\mathfrak{g} = k \oplus m,$$

where $k$ is the Lie algebra of $K$ and $m$ is the orthogonal complement to $k$ in $\mathfrak{g}$ with respect to the Killing form. Restricting the Killing form sign changed to $m$, we get a $G$-invariant Riemannian metric on $M = G/K$ and $\Delta = -C$ (Ikeda and Taniguchi). So using Proposition 2.2 of Ikeda and Taniguchi, the eigenvalues of $\Delta$ can be written for $S^n = SO(n+1)/SO(n)$. We are interested in the Laplacian $\Delta$ acting on 2-forms on $S^{4r-1}$. Let $\tau$ be a Cartan subalgebra of $\mathfrak{g}$ and $\lambda_1, \ldots, \lambda_{2r}$ be linear forms on $\tau$. Any dominant integral form $\Delta$ in $G = SO(4r)$ with respect to $\tau$ is uniquely expressed as

$$\Delta = k_1 \lambda_1 + \ldots + k_{2r} \lambda_{2r},$$

where $k_1, \ldots, k_{2r}$ are integers such that

$$k_1 \geq k_2 \geq \ldots \geq |k_{2r}|.$$

Then using the theorem 4.2 of Ikeda and Taniguchi the following theorem can be easily stated.

**Theorem 1**—If $p = 2$ and $\Lambda_\rho$ is the highest weight of the irreducible representation $\rho$ intervening in $C^\infty(\Lambda^2 S^{4r-1})$, then

$$\Lambda_\rho = (k + 1) \lambda_1 + \lambda_2$$

or

$$\Lambda_\rho = (k + 1) \lambda_1 + \lambda_2 + \lambda_3$$

for $r \geq 2$. 
Moreover with respect to \( \frac{-1}{2(4r-2)} \) times the Killing form, the eigen values of \( \Delta \) are given by

\[
(k + 2)(k + 4r - 2) \quad \text{for} \quad \Lambda_\rho = (k + 1) \lambda_1 + \lambda_2
\]

and

\[
(k + 3)(k + 4r - 3) \quad \text{for} \quad \Lambda_\rho = (k + 1) \lambda_1 + \lambda_2 + \lambda_3.
\]

Here \( k \) runs over all non-negative integers. Moreover the multiplicity \( \mu_\rho \) of the above in \( C^\infty(\Lambda^2 S^{4r-1}) \) is exactly one in the above two cases.

The multiplicity of an eigen value \( \omega \) is the product of the multiplicity \( \mu_\rho \) and the dimension of the representation \( \rho \). The dimension of the representation \( \rho \) can be computed using Weyl's formula:

\[
\begin{align*}
\pi < \Lambda_\rho + \delta, \alpha > & \quad \text{for} \quad \alpha > 0 \\
\pi < \delta, \alpha > & \quad \text{for} \quad \alpha > 0
\end{align*}
\]

Here \( \alpha > 0 \) are the positive roots in \( SO(4r) \) and \( \delta \) is half the sum of the positive roots in \( SO(4r) \).

Hence we get the following table:

<table>
<thead>
<tr>
<th>( \Lambda_\rho )</th>
<th>Eigen value</th>
<th>Multiplicity</th>
</tr>
</thead>
<tbody>
<tr>
<td>((k + 1) \lambda_1 + \lambda_2)</td>
<td>( (k + 2)(k + 4r - 2) )</td>
<td>( \frac{2(k + 1)(k + 4r - 1)(k + 2r)}{(4r - 3)(k + 2)} \left(\frac{k + 4r - 3}{4r - 4}\right) )</td>
</tr>
<tr>
<td>((k + 1) \lambda_1 + \lambda_2 + \lambda_3)</td>
<td>( (k + 3)(k + 4r - 3) )</td>
<td>( \frac{(k + 1)(k + 2r)(k + 4r - 1)(4r - 3)}{(k + 3)(k + 4r - 3)} \left(\frac{k + 4r - 2}{4r - 3}\right) )</td>
</tr>
</tbody>
</table>

Here \( k \) runs over all non-negative integers.

3. **Zeta Function of the Laplacian \( \Delta \) for \( S^{4r-1} \)**

As we consider the positive operator \( \Delta \) acting on 2-forms on \( S^{4r-1} \), we get

\[
\eta(\Delta, s) = \zeta(\Delta, s).
\]

\[
\zeta(\Delta, s) = \sum_{n=0}^{\infty} \frac{2(n + 1)(n + 4r - 1)(n + 2r)}{(4r - 3)(n + 2)} \left(\frac{n + 4r - 3}{4r - 4}\right)
\times \{ (n + 2)(n + 4r - 2) \}^{-s}
\]

(equation continued on p. 410)
\[ + \sum_{n=0}^{\infty} \frac{(n+1)(n+2r)(n+4r-1)4r-3}{(n+3)(n+4r-3)} \times \left(\frac{n+4r-2}{4r-3}\right) \left\{ (n+3)(n+4r-3) \right\}^{-s} \]

we have to compute \( \zeta(\Delta, s) \) at \( s = 0 \) by a method of analytic continuation. We denote this value by \( \zeta(\Delta, 0) \).

We explain below the details of the analytic continuation which we use to compute the spectral invariant of the zeta function for \( S^{4r-1} \).

Let

\[ S = \sum_{n=1}^{\infty} g(n) \{ f(n) \}^{-s}. \]

Here \( g(n) \) and \( f(n) \) are two primitive polynomials of finite degrees in \( n \). Let us assume that the degree of \( f(n) \) be \( k \). Then this series \( S \) has analytic continuation as

\[ \sum_{n=1}^{\infty} g(n) \{ (n^k + P(n))^{-s} - n^{-ks} \} + \sum_{n=1}^{\infty} g(n) n^{-ks} \]

in the entire complex plane.

Here \( f(n) = n^k + P(n) \), \( P(n) \) being a polynomial of degree \( k - 1 \).

Let

\[
\begin{align*}
(I) & = \sum_{n \leq C} g(n) \{ (n^k + P(n))^{-s} - n^{-ks} \} \\
(II) & = \sum_{n > C} g(n) n^{-ks} \left\{ (1 + \frac{P(n)}{n^k})^{-s} - 1 \right\} \\
(III) & = \sum_{n=1}^{\infty} g(n) n^{-ks}.
\end{align*}
\]

Here the positive number \( C \) is chosen such that

\[ | \frac{P(n)}{n^k} | < 1. \]

Then \( S \) has analytic continuation in the entire complex plane with

\[ S = (I) + (II) + (III). \]

(I) is an entire function whose value at \( s = 0 \) is 0. In (II), \( (1 + \frac{P(n)}{n^k})^{-s} \) can be ex-
panded using binomial theorem because \( \left| \frac{P(n)}{n^k} \right| < 1 \). Hence

\[
(\text{II}) = \sum_{n > c} g(n) n^{-ks} \left\{ -s \frac{P(n)}{n^k} \frac{s(s + 1)}{2} \left( \frac{P(n)}{n^k} \right)^2 - \ldots \text{to} \infty \right\}.
\]

As \( s \to 0 \) we have \( s \zeta(s + 1) \to 1 \), \( \zeta \) being the ordinary Riemann zeta function. So when \( S \to 0 \), (II) gives some constants due to first few terms and all the other terms in (II) will tend to zero. Moreover the sum in (II) taken over any finite rectangle tends to zero as \( s \to 0 \). Let \( g(n) \) be a polynomial of degree \( q \) such that

\[
g(n) = \sum_{i=0}^{q} a_i n^i \text{ where } a_i \text{ (for } 0 \leq i \leq q)\]

are constants with \( a_q = 1 \).

Then

\[
(\text{III}) = \sum_{i=0}^{q} a_i \zeta(ks - i).
\]

So when \( s \to 0 \), (III) will contribute some constants to \( \zeta(D, 0) \) and finally we get \( \zeta(D, 0) \).

We now compute the value \( \zeta(D, 0) \) for \( S^{4r-1} \).

Let

\[
\frac{2(n + 1)(n + 4r - 1)(n + 2r)}{(4r - 3)(n + 2)} \left( \frac{n + 4r - 3}{4r - 4} \right) = \frac{2}{(4r - 3)!} g_1(n)
\]

where \( g_1(n) = \sum_{i=0}^{4r-2} a_i n^i \) with \( a_{4r-2} = 1 \). Similarly

we assume that

\[
\frac{(4r - 3)(n + 1)(n + 2r)(n + 4r - 1)}{(n + 3)(n + 4r - 3)} \left( \frac{n + 4r - 2}{4r - 3} \right)
= \frac{1}{(4r - 4)!} g_2(n)
\]

where \( g_2(n) = \sum_{i=0}^{4r-2} b_i n^i \) with \( b_{4r-2} = 1 \).

We also assume that

\[
f_1(n) = (n + 2)(n + 4r - 2)
\]
and 

\[ f_s(n) = (n + 2)(n + 4r - 3). \]

So 

\[ \zeta(\Delta, s) = 2r(4r - 1) + \frac{2r}{3}(4r - 1)(4r - 2) \]

(at \( s = 0 \))

\[ + \left\{ \frac{2}{(4r - 3)} \right\} \left\{ \sum_{n=1}^{\infty} g_1(n) (f_1(n))^{-s} \right\} \]

\[ + \frac{1}{(4r - 4)!} \left\{ \sum_{n=1}^{\infty} g_2(n) (f_2(n))^{-s} \right\} \text{ (at } s = 0). \]

We first find the contribution to \( \zeta(\Delta, 0) \) from

\[ S_1 = \sum_{n=1}^{\infty} g_1(n) (f_1(n))^{-s}. \]

This series has analytic continuation in the entire complex plane as

\[ S_1 = (I) + (II) + (III) \text{ where} \]

\[ (I) = \sum_{n=1}^{C} g_1(n) \left\{ (n^2 + 4rn + 8r - 4)^{-s} - n^{-2s} \right\} \]

\[ (II) = \sum_{n=C+1}^{\infty} g_1(n)n^{-2s} \left\{ (1 + \frac{4rn + 8r - 4}{n^2})^{-s} - 1 \right\} \text{ and} \]

\[ (III) = \sum_{n=1}^{\infty} g_1(n)n^{-2s}. \]

Here \( C \) is chosen such that \( \left| \frac{4rn + 8r - 4}{n^2} \right| < 1. \)

**Remarks:** When \( r = 2(5^7) \), \( C \) can be chosen to be greater or equal to 9. When \( r = 3(5^{11}) \), \( C \) can be chosen to be greater or equal to 13 and so on.

(I) is an entire function whose value at \( s = 0 \) is 0.

\[ (II) = \sum_{n=C+1}^{\infty} g_1(n)n^{-2s} \left\{ -s \cdot \frac{4rn + 8r - 4}{n^2} \right\} \]

*(equation continued on p. 413)*
\[ + \frac{s (s + 1)}{2} \left( \frac{4rn + 8r - 4}{n^2} \right)^2 - \ldots \text{ to } \infty \] .

Let \( P_1 (n) = 4rn + 8r - 4 \) and

\[ g_1 (n) (P_1 (n))^i = \sum_{t=0}^{4r-1} a_{(t, i)} n^t \text{ for } 1 \leq i \leq 4r - 1. \]

Denoting by (II)\( _i \), the \( i \)th term of (II), we get

\[
(II)_1 = -s \sum_{n=C+1}^{\infty} g_1 (n) P_1 (n) n^{-(2s+2)}
\]

\[
(II)_1 \text{ (at } s = 0) = -s \sum_{i=0}^{4r-1} a_{(i, i)} \zeta (2s + 2 - i) (\text{at } s = 0)
\]

\[ = -\frac{1}{2} a_{(1, 1)}. \]

Similarly

\[
(II)_2 \text{ (at } s = 0) = \frac{1}{6} a_{(2, 5)},
\]

\[
(II)_3 \text{ (at } s = 0) = -\frac{1}{5} a_{(3, 5)} \text{ and so on.}
\]

Finally

\[
(II)_{4r-1} \text{ (at } s = 0) = \frac{(-1)^{4r-1}}{2 (4r - 1)} a_{(4r-1, 8r-3)}
\]

and all other terms in (II) will tend to 0. Now

\[
(III) = \sum_{n=1}^{\infty} g_1 (n) n^{-2s}
\]

\[ = \sum_{i=0}^{4r-2} a_i \zeta (2s - i) \]

\[
(III) \text{ (at } s = 0) = -\frac{a_0}{2} - \frac{a_1 B_1}{2} + \frac{a_2 B_2}{4}
\]

\[ - \ldots + \frac{(-1)^{2r-1} a_{4r-3} B_{4r-3}}{4r - 2}. \]

Here \( B_1, B_2, \ldots, B_{2r-1} \) are Bernoulli's numbers\(^8\).

Similarly to find the contribution to \( \zeta (\Delta, 0) \) from

\[ S_1 = \sum_{n=1}^{\infty} g_1 (n) (f_2 (n))^{-i}, \]
we assume that
\[ P_2(n) = 4rn + 12r - 9 \]
and
\[ g_2(n) (P_2(n))^t = \sum_{t=0}^{4r-1} b_{t(1)} n^t \text{ for } 1 \leq t \leq 4r - 1. \]

So using the notations that we have used, the following theorem is completely established.

**Theorem 2**—A spectral invariant of the zeta function \( \zeta(\Delta, s) \) at \( s = 0 \) of the Laplace Beltrami operator \( \Delta \) acting on 2-forms on \( S_{4r-1}^* \) \((r \geq 2)\) is

\[
\frac{2}{(4r - 3)!} \left\{ \sum_{t=1}^{4r-1} \frac{(-1)^t a_{t(2t-1)}}{2t} + \sum_{t=1}^{2r-1} \frac{(-1)^t a_{2t-1} B_t}{2t} - \frac{a_0}{2} \right\}
\]

\[
+ \frac{1}{(4r - 4)!} \left\{ \sum_{t=1}^{4r-1} \frac{(-1)^t b_{t(4t-1)}}{2t} + \sum_{t=1}^{2r-1} \frac{(-1)^t b_{2t-1} B_t}{2t} - \frac{b_0}{2} \right\}
\]

\[
+ \frac{2r}{3} (16r^2 - 1).
\]

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**REFERENCES**