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INCOMPLETE BLOCK DESIGNS OBTAINED BY THE GENERALISED ROW-JUXTAPOSITION AND GENERALISED COLUMN-CONCATENATION OF INCIDENCE MATRICES

G. A. PATWARDHAN

AND

SHAILAJA SHARMA

Department of Mathematics, Indian Institute of Technology, Bombay 400 076

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We define the concepts of 'generalized row-juxtaposition' and 'generalized column-concatenation' of two matrices. By taking these matrices as incidence matrices of block designs, several new constructions of binary and n-ary ($n>2$) incomplete block designs have been obtained.

1. INTRODUCTION

Taking the clue from Helleseth and Van Tilborg[3], we have defined, in the following, the concepts of 'generalized row-juxtaposition' and 'generalized column-concatenation' of matrices. A reference to what we have termed 'generalised column-concatenation' has been made in Helleseth and Van Tilborg[3].

The generalized juxtaposition of the rows and concatenation of the columns of (0, 1)-matrices provide some new constructions of 2-PBIBDS, 3-PBIBDS, partially balanced n-ary designs and balanced block designs. Eleven such constructions have been given in section 3 through various theorems and corollaries.

2. DEFINITIONS

Let A and B be arbitrary matrices of orders $m \times n$ and $p \times q$ respectively, over a field F.

Let

$$\mathbf{r}_i \; ; \; 1 \leqslant i \leqslant m$$

and

$$\mathbf{a}_j \; ; \; 1 \leqslant j \leqslant n$$

denote respectively the row and column vectors of A. Let

$$\mathbf{s}_i \; ; \; 1 \leqslant i \leqslant p$$

and

$$\mathbf{b}_j \; ; \; 1 \leqslant j \leqslant q$$

denote the corresponding vectors of B.

Given the ordered pair (A, B) of matrices, the generalised row-juxtaposition of the pair (A, B), denoted $G(A \mid B)$, is a function that juxtaposes all possible pairs of rows, one each from A and B taken in that order, giving rise to a new matrix $G(A \mid B)$ of order $mp \times (n + q)$:

$$G(A \mid B) = \left(\begin{array}{c|c|c|c} \mathbf{r}_1^T \ \cdots \ \mathbf{r}_1^T & \mathbf{r}_2^T \ \cdots \ \mathbf{r}_2^T & \cdots & \mathbf{r}_m^T \ \cdots \ \mathbf{r}_m^T \\ \mathbf{s}_1^T \ \cdots \ \mathbf{s}_p^T & \mathbf{s}_1^T \ \cdots \ \mathbf{s}_p^T & & \mathbf{s}_1^T \ \cdots \ \mathbf{s}_p^T \end{array} \right)^T.$$

Analogously, given the ordered pair (A, B), the generalized column-concatenation of (A, B), denoted $G(A/B)$ concatenates all possible pairs of columns, one each from A and B taken in that order, giving rise to a new matrix $G(A/B)$ of order $(m + p) \times nq$.

$$G(A/B) = \left(\begin{array}{c|c|c|c} \mathbf{a}_1 \ \cdots \ \mathbf{a}_1 & \mathbf{a}_2 \ \cdots \ \mathbf{a}_2 & \cdots & \mathbf{a}_n \ \cdots \ \mathbf{a}_n \\ \mathbf{b}_1 \ \cdots \ \mathbf{b}_q & \mathbf{b}_1 \ \cdots \ \mathbf{b}_q & & \mathbf{b}_1 \ \cdots \ \mathbf{b}_q \end{array} \right).$$

The following standard abbreviations have been used :

SBIBD	:	symmetric balanced incomplete block design
n-PBIBD	:	n-associate class partially balanced incomplete block design
BBD	:	balanced block design
SRGDD	:	semi-regular group divisible design
S-PBnD	:	S-associate class partially balanced n-ary design
$C_p(A)$	:	Hasse-Minkowski invariant of A
$A(v, b, r, k, \lambda)$	:	BIBD with parameters v, b, r, k, λ having A as its incidence matrix.

3. Main Results

The following theorems can all be proved by the actual construction of the matrices and computation of the design parameters indicated. Most of the proofs are straight-forward, hence omitted.

Theorem 3.1—$G(A \mid B)$, where $A(v_1, b_1, r_1, k_1, \lambda_1)$ and $B(v_2, b_2, r_2, k_2, \lambda_2)$ are BIBDs, is a 3-PBIBD with rectangular association scheme, two unequal block sizes and the following parameters :

$$V = v_1 v_2, \; B = b_1 + b_2, \; R = r_1 + r_2;$$

$$K_1 = k_1 v_2,\ K_2 = k_2 v_1;$$

$$\Lambda_1 = r_1 + \lambda_{12},\ \Lambda_2 = r_2 + \lambda_1,\ \Lambda_3 = \lambda_1 + \lambda_2$$

$$n_1 = v_2 - 1,\ n_2 = v_1 - 1,\ n_3 = (v_1 - 1)(v_2 - 1)$$

there being b_1 blocks of size K_1 and b_2 blocks of size K_2.

Corollary 3.1.1—In the above construction, when $A = B$, $G(A \mid A)$ represents a 2-PBIBD with L_2-association scheme, with parameters

$$V = v^2,\ B = 2b,\ R = 2r,\ K = vk;$$

$$\Lambda_1 = r + \lambda,\ \Lambda_2 = 2\lambda;$$

$$n_1 = 2(v - 1),\ n_2 = (v - 1)^2.$$

Raghavarao[4] (p. 159) has given a construction for a 2-PBIBD with the above parameters, wherein he is required to use a BIBD and also an association scheme of Latin square type for constructing this 2-PBIBD. Our construction requires only a BIBD $A(v, b, r, k, \lambda)$ and nothing else.

Corollary 3.1.2—Each submatrix of $G(A \mid B)$ with constant column sum constitutes a 2-PBIBD of the singular group divisible type. The parameters of these designs are

$$(v_1 v_2,\ b_1,\ r_1,\ v_2 k_1\ ;\ r_1,\ \lambda_1\ ;\ v_2 - 1,\ (v_1 - 1) v_2)$$

and

$$(v_1 v_2,\ b_2,\ r_2,\ v_1 k_2\ ;\ r_2,\ \lambda_2\ ;\ v_1 - 1,\ (v_2 - 1) v_1)$$

respectively.

Since

$$G(A^T \mid B^T)^T = G(A/B),$$

we also have

Corollary 3.1.3—The transpose of $G(A/B)$ is a 3-PBIBD with rectangular association scheme and unequal block sizes, provided that $A(v_1, b_1, r_1, k_1, \lambda_1)$ and $B(v_2, b_2, r_2, k_2, \lambda_2)$ are linked block designs with respective block intersection numbers μ_1 and μ_2. Its parameters are

$$V = b_1 b_2,\ B = v_1 + v_2,\ R = k_1 + k_2\ ;\ K_1 = b_2 r_1,\ K_2 = b_1 r_2$$

$$\Lambda_1 = k_1 + \mu_2,\ \Lambda_2 = k_2 + \mu_1,\ \Lambda_3 = \mu_1 + \mu_2\ ;\ n_1 = b_2 - 1$$

$$n_2 = b_1 - 1,\ n_3 = (b_1 - 1)(b_2 - 1).$$

Corollary 3.1.4—If $A = B$ in Corollary 3.1.3, then a 2-PBIBD with L_2-association scheme can always be constructed, whose parameters are $V = b^2$, $B = 2v$, $R = 2k$, $K = br$; $\Lambda_1 = k + \mu$, $\Lambda_2 = 2\mu$; $n_1 = 2(b - 1)$, $n_2 = (b - 1)^2$.

Theorem 3.2—If $A(v_1, b_1, r_1, k_1, \lambda_1)$ and $B(v_2, b_2, r_2, k_2, \lambda_2)$ are BIBDS, then $G(A/B)$ is a BBD with two different numbers of replicates as defined by Agrawal[1]. Its parameters are

$$V = v_1 + v_2,\ B = b_1 b_2;\ R_1 = b_2 r_1,\ R_2 = b_1 r_2$$

$$K = k_1 + k_2;\ \Lambda_1 = b_2 \lambda_1,\ \Lambda_2 = b_1 \lambda_2,\ \Lambda_{12} = r_1 r_2.$$

This result is corroborated by the structure of $G(A/B)$. $[G(A/B)]^T$, which is

$$\left[\begin{array}{c|c} b_2 \cdot AA^T & r_1 r_2 \cdot E_{v_1 v_2} \\ \hline r_1 r_2 \cdot E_{v_2 v_1} & b_1 \cdot BB^T \end{array}\right].$$

Corollary 3.2.1—When $A = B$ in the above theorem, the BBD becomes a SRGDD with parameters

$$V = 2v,\ B = b^2,\ R = rb,\ K = 2k$$

$$\Lambda_1 = b\lambda,\ \Lambda_2 = r^2;\ n_1 = v - 1,\ n_2 = v.$$

Corollary 3.2.2—The transpose of $G(A \mid B)$ is a BBD with two different numbers of replicates, provided that A and B are both linked block designs with respective block intersection numbers μ_1 and μ_2. Its parameters are

$$V = b_1 + b_2,\ B = v_1 v_2;\ R_1 = v_2 k_1;\ R_2 = v_1 k_2;\ K = r_1 + r_2$$

$$\Lambda_1 = v_2 \mu_1,\ \Lambda_2 = v_1 \mu_2,\ \Lambda_{12} = k_1 k_2.$$

This result is corroborated by the form of the matrix $[G(A \mid B)]^T$. $G(A \mid B)$, which is

$$\left[\begin{array}{c|c} v_2 \cdot A^TA & k_1 k_2 E_{b_1 b_2} \\ \hline k_1 k_2 \cdot E_{b_2 b_1} & v_1 \cdot B^TB \end{array}\right].$$

Corollary 3.2.3—If $A = B$ in Corollary 3.2.2, $[G(A \mid A)]^T$ represents a SRGDD with parameters

$$V = 2b,\ B = v^2,\ R = vk,\ K = 2r;\ \Lambda_1 = v\mu,\ \Lambda_2 = k^2$$

$$n_1 = v - 1,\ n_2 = v.$$

If $A_{m,n}$ and $B_{p,q}$ are any two matrices over F, then the product $G(A \mid B)$ $G(A/B)$ is defined if and only if $m + p = n + q$ and the product $G(A/B)$ $G(A \mid B)$ is defined if and only if

$$mp = nq.$$

If A and B are the incidence matrices of BIBDs, then a necessary and sufficient condition for the existence of any or both of the above product—matrices is that A and B are SBIBDSs.

In what follows, unless stated otherwise, $A\ (v_1, k_1, \lambda_1)$ and $B\ (v_2, k_2, \lambda_2)$ will be assumed to be SBIBDs.

It has been shown[5] that the Kronecker sum of incomplete block designs gives ternary designs. We obtain below, an n-ary design as the Kronecker sum of the squares of two SBIBDs.

Theorem 3.3—The product

$$G\ (A \mid B)\ G\ (A/B)$$

is the Kronecker sum of A^2 and B^2 and is the incidence matrix of a 3-PBnD with rectangular association scheme, where $n = k_1 + k_2 + 1$. The parameters of the design are

$$V = v_1\, v_2 = B,\quad R = v_1\, k_2^2 + v_2\, k_1^2 = K$$

$$\Lambda_1 = v_2\, k_1\, ((k_1 - 1)\, \lambda_1 + k_1) + v_1\, \lambda_2\, (k_2^2 + k_2 - \lambda_2) + 2k_1^2\, k_2^2$$

$$\Lambda_2 = v_2\, \lambda_1\, (k_1^2 + k_1 - \lambda_1) + v_1\, k_2\, ((k_2 - 1)\, \lambda_2 + k_2) + 2k_1^2\, k_2^2$$

$$\Lambda_3 = v_2\, \lambda_1\, (k_1^2 + k_1 - \lambda_1) + v_1\, \lambda_2\, (k_2^2 + k_2 - \lambda_2) + 2k_1^2\, k_2^2$$

$$n_1 = v_2 - 1,\ n_2 = v_1 - 1,\ n_3 = (v_1 - 1)\, (v_2 - 1).$$

In the above theorem, as well as in the ones that follow, we have extended the definition of n-ary designs given by Das and Rao[3] to include those designs for which incidence matrices have any of n distinct non-negative integers as its elements. The entries in the matrix $G\ (A \mid B)\ G\ (A/B)$ above belong to the set $\{0, 1, \ldots, k_1 + k_2\}$.

Corollary 3.3.1—In the above theorem, if $A = B$, then a 2-PBnD can always be constructed, where $n = 2k + 1$. Its parameters are

$$V = v^2 = B,\ R = 2vk^2 = K$$

$$\Lambda_1 = v\lambda\, (k^2 + k - \lambda) + vk^3 + 2k^4$$

$$\Lambda_2 = 2v\, \lambda\, (k^2 + k - \lambda) + 2k^4$$

$$n_1 = 2\, (v - 1),\ n_2 = (v - 1)^2.$$

The above design has L_2-association scheme.

Agrawal[1] has defined binary BBDs with constant block sizes. In the following theorem, we obtain an n-ary BBD with unequal block sizes.

Theorem 3.4—The matrix $G\ (A/B)\ G\ (A \mid B)$

$$= \left[\begin{array}{c|c} \dfrac{v_2 A^2}{k_1 k_2 E_{\substack{v \, v \\ 2 \, 1}}} & \dfrac{k_1 k_2 E_{\substack{v \, v \\ 1 \, 2}}}{v_1 B^2} \end{array} \right]$$

is the incidence matrix of an n-ary BBD with two different numbers of replications and two different block sizes. The parameters of this design are

$$V = v_1 + v_2 = B$$

$$R_1 = v_2 k_1 (k_1 + k_2) = K_1$$

$$R_2 = v_1 k_2 (k_1 + k_2) = K_2$$

$$\Lambda_1 = \lambda_1 v_2^2 (k_1^2 + k_1 - \lambda_1) + k_1^2 k_2^2 v_2$$

$$\Lambda_2 = \lambda_2 v_1^2 (k_2^2 + k_2 - \lambda_2) + k_1^2 k_2^2 v_1$$

$$\Lambda_{12} = k_1 k_2 (v_2 k_1^2 + v_1 k_2^2).$$

Here

$$n = k_1 + k_2 + 2.$$

Corollary 3.4.1—In the above theorem, replacing B by A we get a group divisible 2-PBnD, where $n = k + 2$, with the following parameters :

$$V = 2v = B, \; R = 2vk^2 = K$$

$$\Lambda_1 = \lambda v^2 (k^2 + k - \lambda) + k^4 v, \; \Lambda_2 = 2vk^4$$

$$n_1 = v - 1, \; n_2 = v.$$

Theorem 3.5—If A, B, C and D are four linked BIBDs with parameters respectively

$$(v_i, b_i, r_i, k_i, \lambda_i, \mu_i), \; i = 1, 2, 3, 4$$

such that

$$v_1 = b_3$$

and

$$v_2 = b_4$$

then

$$E = [G(A \mid B) \mid [G(C/D)]^T]$$

is a rectangular 3-PBIBD with four unequal block sizes. Its parameters are

$$V = v_1 v_2, \; B = 2(v_1 + v_2), \; R = r_1 + r_2 + k_3 + k_4$$

$$K_1 = v_2 k_1,\ K_2 = v_1 k_2,\ K_3 = v_2 r_3,\ K_4 = v_1 r_4$$

$$\Lambda_1 = r_1 + \lambda_2 + k_3 + \mu_4,\ \Lambda_2 = \lambda_1 + r_2 + \mu_3 + k_4,$$

$$\Lambda_3 = \lambda_1 + \lambda_2 + \mu_3 + \mu_4$$

$$n_1 = v_2 - 1,\ n_2 = v_1 - 1,\ n_3 = (v_1 - 1)(v_2 - 1).$$

Corollary 3.5.1—Under the condition that $A = B = C = D$, the matrix $F = E^T$ (where E is as given in Theorem 3.5) is a 3-PBIBD with Extended Group divisible association scheme and parameters given by

$$V = 4v,\ B = v^2,\ R = kv,\ K = 4k$$

$$\Lambda_1 = v\lambda,\ \Lambda_2 = k^2,\ \Lambda_3 = vk$$

$$n_1 = 2(v - 1),\ n_2 = 2v,\ n_3 = 1.$$

Lemma 3.1—Suppose that A, B, C and D are BIBDs with respective sets of parameters

$$(v_i, b_i, r_i, k_i, \lambda_i),\ i = 1, 2, 3, 4.$$

(i) If $N = G(A \mid B)$, then $NN^T = AA^T \circledcirc BB^T$

(ii) If $N = [G(A/B)]^T$, then $NN^T = A^TA \circledcirc B^TB$

(iii) If $N = [G(A \mid B) \mid [G(A/B)]^T]$

then

$$NN^T = (AA^T + C^TC) \circledcirc (BB^T + D^TD$$

when

$$v_1 = b_3 \text{ and } v_2 = b_4.$$

where $\circledcirc$ denote the Kronecker sum of matrices.

The above lemma may be useful in determining $C_p(NN^T)$ and hence finding necessary conditions for the existence of the designs denoted by N.

4. Summary

The following table summarizes the results obtained in the last section.

Sr. No.	Source	Operation	Result
1.	Thm. 3.1	$G(A \mid B)$	3-PBIBD with rect. asso. Sch. and 2 unequal block sizes.
2.	Cor. 3.1.1	$G(A \mid A)$	2-PBIBD with L_2 asso. sch.

(Table continued on p. 422)

Sr. No.	Source	Operation	Result
3.	Cor. 3.1.2	Submatrix of $G(A \mid B)$	Singular GDD
4.	Thm. 3.2	$G(A/B)$	BBD with two replicates.
5.	Cor. 3.2.1	$G(A/A)$	SRGDD
6.	Thm. 3.3	$G(A \mid B)\, G(A/B) = A^2 \circledcirc B^2$	3-PBnD with rect. asso. sch.
7.	Cor. 3.3.1	$G(A \mid A)\, G(A/A)$	2-PBnD with L_2-asso. sch.
8.	Thm. 3.4	$G(A/B)\, G(A \mid B)$	n-ary BBD with two replicates
9.	Cor. 3.4.1	$G(A/A)\, G(A \mid A)$	GD-PBnD.
10.	Thm. 3.5	$\{G(A \mid B) \mid [G(C/D)]^T\}$	3-PBIBD with rect. asso. sch. and 4 block sizes.
11.	Cor. 3.5.1	$\{G(A \mid A) \mid [G(A/A)]^T\}$	3-PBIBD with extended GD asso. Sch.

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