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ON *C* 3-LIKE FINSLER SPACES

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In this paper we have studied the properties of Finsler space F^n of dimension $n \geqslant 4$ in which C_{ijk} is of the form $C_{ijk} = \mathscr{C}_{(ijk)} \{h_{ij} a_k + C_i C_j b_k\}$. Since C_{ijk} of any three dimensional Finsler space is of the above form, any n-dimensional Finsler space F^n ($n \geqslant 4$) will be called a $C3$-like Finsler space if its C_{ijk} is of this from.

1. INTRODUCTION

There are three kinds of torsion tensor in Cartan's theory of Finsler space F^n. Two of them are (*h*) *hv*-torsion tensor C_{ijk} and (*v*) *hv*-torsion tensor P_{ijk} which are symmetric in all its indices. It is obvious that F^n is Riemannian if the tensor C_{ijk} vanishes and Deicke's theorem shows that F^n is also Riemannian even if the tensor C_i ($=g^{jk} C_{ijk}$), contracted by the reciprocal tensor g^{jk} of the fundamental tensor g_{jk}, vanishes[3,5,14,17]. As is well known, in the 2-dimensional Finsler space F^2, C_{ijk} is always written in the form

$$LC_{ijk} = Im_i m_j m_k \qquad \ldots(1.1)$$

where $L(x,y)$ is the fundamental function, m_i is unit vector orthogonal to the element of support y^i and I is called the principal scalar by Berwald[1,2,17]. A Moor studied the property of n-dimensional Finsler space F^n ($n \geqslant 3$) in which C_{ijk} is of the form (1.1)[12,13]. In this case, however the third curvature tensor S_{ijkl} vanishes, so that F^n is reduced to a Riemannian space in virture of Brickell's theorem provided that the metric is symmetric[4,18].

In a 3-dimensional Finsler space F^3, C_{ijk} is always written in the form

$$L\,C_{ijk} = Hm_i m_j m_k - J\,\mathscr{C}_{(ijk)} \{m_i m_j n_k\} + I\,\mathscr{C}_{(ijk)} \{m_i n_j n_k\} + J\,n_i n_j n_k \qquad \ldots(1.2)$$

where $\mathscr{C}_{(ijk)}\{\ \}$ denote the cyclic permutation of indices i, j, k and addition. H, I and J are main scalars and (l_i, m_i, n_i) is Moor's frame[7,9,15]. Here $l_i = \dot{\partial}_i L$ is the unit vector along the element of support, m_i is the unit vector along C_i i.e. $m_i = C_i/C$ where $C^2 = g^{ij} C_i C_j$ and n_i is a unit vector orthogonal to the vectors l_i and m_i. Since the angular metric tensor h_{ij} in F^3 can be written as

$$h_{ij} = m_i m_j + n_i n_j, \qquad \ldots(1.3)$$

we may write (1.2) as

$$C_{ijk} = \mathscr{C}_{(ijk)}\{h_{ij}\, a_k + C_i C_j b_k\} \qquad \ldots(1.4)$$

where

$$a_k = \frac{1}{L}\left(I\, m_k + \frac{J}{3} n_k \right)$$

and

$$b_k = \frac{1}{LC^2}\left\{\left(\frac{H}{3} - I \right) m_k + \frac{4J}{3} n_k \right\}.$$

Many workers[6,10,11,16] have obtained various interesting special forms of C_{ijk}. They are

Generalized C-reducible :

$$C_{ijk} = \mathscr{C}_{(ijk)}\{P_{ij}\ Q_k\}. \qquad \ldots(1.5)$$

Quasi C-reducible :

$$C_{ijk} = \mathscr{C}_{(ijk)}\{P_{ij}\ C_k\} \qquad \ldots(1.6)$$

Semi C-reducible :

$$C_{ijk} = \frac{P}{n+1}\ \mathscr{C}_{(ijk)}\left\{h_{ij}\, C_k\right\} + \frac{q}{C^2}\, C_i C_j C_k \qquad \ldots(1.7)$$

C-reducible :

$$C_{ijk} = \frac{1}{n+1}\, \mathscr{C}_{(ijk)}\{h_{ij}\, C_k\} \qquad \ldots(1.8)$$

$C2$-like :

$$C_{ijk} = \frac{1}{C^2}\, C_i C_j C_k \qquad \ldots(1.9)$$

where P_{ij} in (1.5) and (1.6) is a symmetric tensor, Q_i in (1.5) is a covariant vector such that $P_{i0} = Q_0 = 0$ and p, q in (1.7) are scalars satisfying $p + q = 1$.

The purpose of this paper is to study the properties of F^n of dimension $n \geqslant 4$ in which C_{ijk} is of the form (1.4). Since C_{ijk} of any three dimensional Finsler space is

of the form (1.4), an n-dimensional Finsler space F^n ($n \geqslant 4$) will be called a $C3$-like Finsler space if its C_{ijk} is of the form (1.4).

2. The Properties of C3-like Finsler Spaces

Definition 2.1— A Finsler space F^n of dimension $n \geqslant 4$ is called a $C3$-like Finsler space if there exist covariant vector fields a_k and b_k in F^n such that its (h) hv-torsion tensor C_{ijk} can be written as

$$C_{ijk} = \mathscr{C}_{(ijk)} \{h_{ij} a_k + C_i C_j b_k\}. \qquad ...(2.1)$$

Since C_{ijk} is an indicatory tensor it follows that a_k and b_k are indicatory tensor i.e. $a_0 = b_0 = 0$. If b_k is a null vector then contracting (2.1) with g^{jk} and putting $b_k = 0$ we get $a_k = \frac{1}{n+1} C_k$. Hence it reduces to a C-reducible Finsler space. Thus a $C3$-like Finsler space is a generalization of a C-reducible Finsler space. Again if a_k is a null vector then contraction of (2.1) with g^{jk} gives

$$C_i = C^2 b_i + 2 b_j C^j C_i \qquad ...(2.2)$$

which after contraction with C^i gives $b_j C^j = \frac{1}{3}$. Thus (2.2) gives $b_i = \frac{C_i}{3C^2}$. Hence for $a_k = 0$, the form (2.1) of C_{ijk} reduce to

$$C_{ijk} = \frac{1}{C^2} C_i C_j C_k.$$

Therefore the $C3$-like Finsler space is also a generalization of $C2$-like Finsler space. Furthermore if we put $a_k = PC_k/n+1$, and $b_k = qC_k/3C^4$ then the form (2.1) reduces to the form (1.7) i.e. F^n becomes a semi C-reducible Finsler space. Thus a $C3$-like Finsler space is also a generalization of semi-C-reducible Finsler space. Contracting (2.1) with g^{jk} we get

$$KC_i = (n + 1) a_i + C^2 b_i \qquad ...(2.3)$$

where $K = 1 - b_j c^j$. If $b_j c^j = 1$, then $K = 0$ and hence from (2.3) we get $a_i = -\frac{C^2}{n+1} b_i$. Substituting this value of a_i in (2.1) we get

$$C_{ijk} = \mathscr{C}_{(ijk)} \{P_{ij} b_k\} \qquad ...(2.4)$$

where

$$P_{ij} = C_i C_j - \frac{C^2}{n+1} h_{ij}.$$

The form (2.4) is the form (1.5). A Finsler space whose C_{ijk} is of the form (1.5) is called generalized C-reducible Finsler space[16]. The examples of non-Riemannian generalized C-reducible Finsler space is given by Okada and Numata[16]. If

$$\alpha = \{a_{ij}(x) y^i y^j\}^{1/2}$$

and

$$\beta = \{b_{ij}(x)\, y^i y^j\}^{1/2}$$

be two independent functions then an n ($\geqslant 2$) dimensional Finsler space F^n with a metric $L = L(\alpha, \beta)$ is generalized C-reducile.

Hence we have the following :

Theorem 2.1—Every $C3$-like Finsler space is a generalized C-reducible Finsler space provided $b_i C^i = 1$.

If $b_i C^i \neq 1$ then from (2.1) and (2.3) we get

$$C_{ijk} = \mathscr{C}_{(ijk)} \{Q_{ij} a_k\} + d_i d_j d_k \qquad \ldots(2.5)$$

where

$$Q_{ij} = h_{ij} + \tfrac{1}{2}\left(\frac{n+1}{K}\right)^2 (a_i b_j + a_j b_i) + \frac{2(n+1)C^2}{K^2} b_i b_j \qquad \ldots(2.6)$$

and

$$d_i = \left(\frac{3C^4}{K^2}\right)^{1/3} b_i.$$

Now we shall give below an example of a Finsler space whose (h) hv-torsion tensor C_{ijk} is of the form (2.5). Let F^n be a C-reducible Finsler space with metric function $L(x, y)$. If F^{*n} be another Finsler space whose metric function $L^*(x, y)$ is given by

$$L^{*3}(x, y) = L^3(x, y) + (b_i y^i)^3$$

where $b_i(x)$ is one form in F^n. The (h) hv-torsion tensor C^*_{ijk} of F^{*n} and those of F^n are related by

$$C^*_{ijk} = P C_{ijk} + \frac{p q^2}{2L}(h_{ij} m_k + h_{ik} m_j + h_{jk} m_i) - \frac{2p^3 - 1}{L} p\, m_i m_j m_k \qquad \ldots(2.7)$$

where

$$m_i = q l_i - p b_i,\ q = \frac{L}{L^*},\ q = \frac{\beta}{L^*},\ \beta = b_i y^i.$$

Thus putting

$$C_{ijk} = \frac{1}{n+1}(h_{ij} C_k + h_{ik} C_j + h_{jk} C_i) \text{ in (2.7)}$$

we get

$$C^*_{ijk} = \mathscr{C}_{(ijk)} \{Q_{ij} a_k\} + d_i d_j d_k$$

where

$$Q_{ij} = p\, h_{ij},\ a_i = \frac{C_i}{n+1} + \frac{q^2}{2L} m_i \text{ and } d_i = -\left(\frac{2p^3 - 1}{L}\, p\right)^{\frac{1}{3}} m_i.$$

A Randers space is a Finsler space whose metric funciion $L(x, y)$ is of the form

$$L(x, y) = (a_{ij}(x)\, y^i y^j)^{1/2} + f_i(x)\, y^i$$

where $a_{ij}(x)$ is a Riemannian metric tensor and $f_i(x)$ is one form.

A Kropina space is a Finsler space in which

$$L(x, y) = \frac{(a_{ij}(x)\, y^i y^j)^{1/2}}{f_i(x)\, y^i}.$$

It is well known[8,19] that the Randers space and the Kropina space are C-reducible Finsler spaces. Thus if the metric function $L^*(x, y)$ of F^{*n} is given by

$$L^{*3}(x, y) = \{(a_{ij}(x)\, y^i y^j)^{1/2} + f_i(x)\, y^i\}^3 + (b_i\, y^i)^3$$

or

$$L^{*2}(x, y) = \frac{(a_{ij}(x)\, y^i\, y^j)^{3/2}}{(f_i(x)\, y^i)^3} + (b_i\, y^i)^3$$

then the (h) hv-torsion tensor of F^{*n} is of the form (2.5).

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