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A NOTE ON THE CHANDRASEKHAR'S X-FUNCTION

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A differentiability property of the Chandrasekhar's X-Function is analyzed.

INTRODUCTION

Chandrasekhar's X and Y Function play a vital role in the theory of radiative transfer[1] and neutron transport theory[2]. These functions are defined as follows :

$$\left.\begin{aligned} X(z) &= \lim_{x \to 0^+} J(x, z) \\ \text{and} \\ Y(z) &= \lim_{x \to t^-} J(x, z) \end{aligned}\right\} \quad \ldots(1.1)$$

where $J(x, z)$ is the unique solution of the integral equation

$$J(x, z) = \exp(ixz) + \int_0^t K(x-y)\, J(y, z)\, dy. \quad \ldots(1.2)$$

We denote the functions $X(z)$, $Y(z)$ and $J(x, z)$ by $X(z, t)$, $Y(z, t)$ and $J(x, z, t)$, respectively, to emphasize their dependence on t. In this note we analyse differentiability property of the X-function with respect to t.

Let T_t be an integral operator from $L_2[0, t]$ into itself defined as

$$T_t(f) = \int_0^t K(x - y)\, f(y)\, dy.$$

The adjoint operator T_t^* defined by

$$T_t^*(f) = \int_0^t \overline{K(y - x)}\, f(y)\, d(y).$$

Assuming that T_t is real and symmetric, i. e.

$$K(x) \text{ is real and } K(x) = K(-x) \quad \ldots(1.3)$$

$$K(x) \text{ is locally square integrable} \quad \ldots(1.4)$$

$$K(x) \in L_1(-\infty, \infty) \qquad \ldots(1.5)$$

and

$$1 - \hat{K}(\omega) \neq 0, \quad -\infty < \omega < \infty \qquad \ldots(1.6)$$

where $\hat{K}(\omega)$ denotes the Fourier transform of $K(x)$. Vittal Rao[3] proved that

$$\frac{d}{dt} X(0, t) = \alpha(t, t)\, X(0, t) \qquad \ldots(1.7)$$

where $\alpha(x, t)$ is the unique solution of the equation

$$\alpha(x, t) = K(x) + \int_0^t K(x - y)\, \alpha(y, t)\, dy. \qquad \ldots(1.8)$$

In this note we generalize this result for a larger class of operators, which satisfy, instead of (1.3) and (1.6) the following conditions

$$T_t \text{ is normal for all } t \qquad \ldots(1.3)'$$

$$1 - |\hat{K}(\omega)| \neq 0 \quad -\infty < \omega < \infty. \qquad \ldots(1.6)'$$

The generalization is established by the following theorem.

Theorem 1—The unique solution $\alpha(x, t)$ of (1.8) is differentiable with respect to t for almost every t and its derivative is $\alpha(t, t)\, \alpha_1(x = t, t)$ in the following sense.

$$\lim_{h \to 0} \int_0^t \left| \frac{\alpha(x, t + h) - \alpha(x, t)}{h} - \alpha(t, t)\, \alpha_1(x - t, t) \right| dx = 0 \qquad \ldots(1.9)$$

where $\alpha_1(x, t)$ is the unique solution of the integral equation

$$\alpha_1(x, t) = K(x) + \int_{-t}^{0} K(x - y)\, \alpha_1(y, t)\, dy. \qquad \ldots(1.10)$$

Theorem 2—For real z

$$\frac{\partial}{\partial t} X(z, t) = \exp(itz)\, \overline{\alpha^*(t, t)}\, X^*(z, t) \qquad \ldots(1.11)$$

$$\frac{\partial}{\partial t} X^*(z, t) = \exp(-itz)\, \overline{\alpha(t, t)}\, X(z, t) \qquad \ldots(1.12)$$

where $X^*(z, t) = \lim_{x \to 0^+} J^*(x, z, t)$, $J^*(x, z, t)$ is the unique solution of the equation

$$J^*(x, z, t) = \exp(-ixz) + \int_0^t K(y - x)\, J^*(y, z, t)\, dy \qquad \ldots(1.13)$$

and $\alpha^*(x, t)$ is the adjoint of $\alpha^*(x, t)$ given by

$$\alpha^*(x, y) = \overline{K(-x)} + \int_0^t \overline{K(y-x)}\, \alpha^*(y, t)\, dy. \qquad \ldots(1.14)$$

Proof of the Theorems

First we note that 1 is not an eigenvalue of the operator T and hence $(1 - T_t)^{-1}$ exists.

Proof of the Theorem 1—From (1.10) we get

$$\alpha_1(x - t, t) = K(x - t) + \int_{-t}^{0} K(x - t - y)\, \alpha_1(y, t)\, dy$$

$$= K(x - t) + \int_0^t K(x - s)\, \alpha_1(s - t, t)\, ds. \qquad \ldots(2.1)$$

Hence

$$\alpha_1(x - t, t) = (1 - T_t)^{-1}(K(x - t)). \qquad \ldots(2.2)$$

Now

$$\frac{\alpha(x, t) - \alpha(x, t - h)}{h} = \int_0^{t-h} K(x - y)\, \frac{\alpha(y, t) - \alpha(y, t - h)}{h}$$

$$+ \frac{1}{h} \int_{t-h}^{t} K(x - y)\, \alpha(y, t)\, dy.$$

Hence

$$\frac{\alpha(x, t) - \alpha(x, t - h)}{h} = (1 - T_{t-h})^{-1} \left[\frac{1}{h} \int_{t-h}^{t} K(x - y)\, \alpha(y, t)\, dy \right]. \qquad \ldots(2.3)$$

Define

$$D_h(x) = \frac{\alpha(x, t) - \alpha(x, t - h)}{h} - \alpha(t, t)\, \alpha_1(x - t, t).$$

Using (2.2) and (2.3) it can be easily verified as in Theorem 1 of Vittal Rao[3] that

$$\underset{h \to 0}{Lt} \| D_h(x) \|_1 = \underset{h \to 0}{Lt} \int_0^t \left| \frac{\alpha(x, t) - \alpha(x, t - h)}{h} \right.$$

$$\left. - \alpha(t, t)\, \alpha_1(x - t, t) \right| dx = 0.$$

Similarly (1.9) can also be proved. This completes the proof.

Proof of the Theorem 2—Let R_t^* be the resolvent of the operator T_t^*. Then

$$R_t^*(x, y) - \overline{K(y-x)} = \int_0^t R_t^*(x, s)\, \overline{K(y-s)}\, ds. \qquad \ldots(2.4)$$

From (2.4) an (1.8) it follows that

$$R_t^*(0, y) = \overline{\alpha(y, t)}\ . \qquad \ldots(2.5)$$

The solution $J^*(x, z, t)$ of eqn. (1.13) is given by

$$J^*(x, z, t) = \exp(-ixz) + \int_0^t R_t^*(x, y) \exp(-iyz)\, dy. \qquad \ldots(2.6)$$

Therefore

$$X^*(z, t) = \operatorname*{Lim}_{x \to 0^+} J^*(x, z, t)$$

$$= 1 + \int_0^t \overline{\alpha(y, t)} \exp(-iyz)\, dy. \qquad \text{(From (2.5))}$$

Similarly

$$X(z, t) = 1 + \int_0^t \alpha^*(y, t) \exp(iyz)\, dy. \qquad \ldots(2.7)$$

Now

$$\frac{X^*(z, t) - X^*(z, t-h)}{h} = \int_0^{t-h} \frac{\overline{\alpha(y, t)} - \overline{\alpha(y, t-h)}}{h}$$

$$\exp(-iyz)\, dy + \frac{1}{h} \int_{t-h}^{t} \alpha(y, t) \exp(-iyz)\, dy. \qquad \ldots(2.8)$$

From Theorem 1 it follows that for real z the first term in the right-hand side of (2.8) converges to $\overline{\alpha(t, t)} \int_0^t \overline{\alpha_1(y-t, t)} \exp(-iyz)\, dy$ and the second term converges to $\overline{\alpha(t, t)} \exp(-itz)$ as $h \to 0$.

Hence

$$\frac{\partial}{\partial t} X^*(z, t) = \overline{\alpha(t, t)} \left[\exp(-itz) + \int_0^t \overline{\alpha_1(y - t, t)} \exp(-iyz)\, dy \right].$$

Substituting $y = t - x$ and observing that

$$\overline{\alpha_1(-x, t)} = \alpha^*(x, t)$$

we get

$$\frac{\partial}{\partial t} X^*(z, t) = \overline{\alpha(t, t)} \exp(-itz) \left[1 + \int_0^t \alpha^*(x, t) \exp(ixz)\, dx \right].$$

This gives the formula (1.12). Similarly (1.11) can be proved.

A relation between the X-function and the eigenvalues had been obtained for symmetric integral operators[3]. Many spectral properties of symmetric operators had been extended for normal operators by Vittal Rao and and Sukavanam[4]. The results of this paper may be of use in obtaining a relation between eigenvalues and the X-function for normal operators.

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