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DIFFRACTION OF LOVE WAVES BY TWO PARALLEL PERFECTLY WEAK HALF PLANES

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We consider the diffraction of Love waves by two parallel perfectly weak half planes in a layer overlying a half space. The problem is formulated in terms of the Wiener-Hopf equations in the transformed plane. The transmitted waves are then calculated using the Wiener-Hopf procedure and inverse transforms.

1. INTRODUCTION

The diffraction of seismic waves by structural dicontinuities is of considerable im portance in seismology because of the existence of such discontinuities in the Earth's crust. Exact analytical solutions of these problems are difficult to obtain even for simple geometries. de Hoop[2] presented a method based upon the Wiener-Hopf technique for the solution of bcdy waves by a single perfectly rigid or perfectly weak half plane. Kazi[4] considered the diffraction of Love-waves by perfectly rigid and perfectly weak half planes lying in a surface layer overlying a half space. Recently, Asghar and Zaman[1] have considered the diffraction of Love waves by taking the rigid barrier to be of finite extension.

In this paper, we set up and solve the problem of diffraction of Love waves normally incident on two parallel perfectly weak (crack) half planes lying in a surface layer and parallel to the interface between the layer and the half space. The problem is formulated in terms of the two Wiener-Hopf equations and can be solved by the technique introduced by Jones[3]. The weak screens separate the layer into three loosely coupled layers. The transmitted waves in these three regions have been calculated analytically. As expected on physical grounds, it has been shown that the transmitted wave in each region satisfies the dispersion relation of the Love waves travelling in a layer of uniform thickness under similar boundary condition.

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2. Formulation of the Problem and the Wiener-Hopf Equations

We consider the diffraction of Love waves by two parallel weak half planes (cracks) lying in a layer of uniform thickness H over an elastic half space. The half space has a rigidity μ_1 and shear wave velocity β_1 and the layer has rigidity μ_2 and the shear wave velocity β_2. The coordinate system is chosen in such a way that the interface between the half space and the layered medium conicides with the xy plane, the z axis is directed into the half space and the two semi-infinite planes occupy $z = -h_1, x < 0$ and $z = -h_2, x < 0$. The free surface is $z = -H$. The geometry of the problem is shown in Fig. 1.

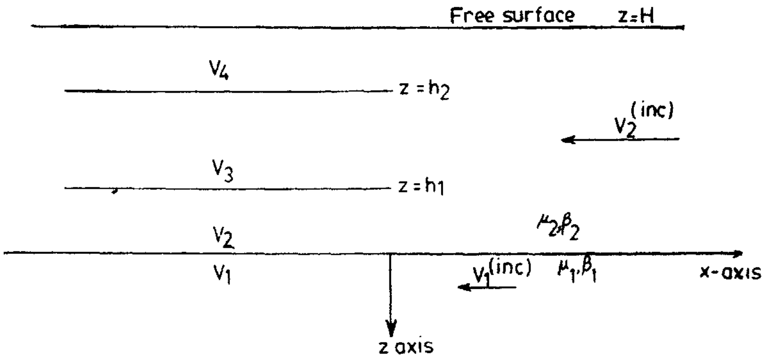

Fig. 1.

Suppressing the time dependence $e^{i\omega t}$, the incident Love waves of the Nth mode have the displacements :

$$V_1^{inc} = A \cos(\sigma_{2N} H) \exp\{-\sigma_{1N} Z - iK_{1N}(x - x0)\},\ Z > 0$$

$$V_2^{inc} = A \cos\{(Z + H)\sigma_{2N}\} \exp\{-iK_{1N}(x - x0)\},\ 0 > Z > -H \quad \ldots(1)$$

where

$$\sigma_{1N} = (K_{1N} - k_1^2)^{1/2},\ \sigma_{2N} = (k_2^2 - K_{1N}^2)^{1/2},\ \frac{\omega}{\beta_1},\quad k_2 = \frac{\omega}{\beta_2} \quad \ldots(2)$$

and K_{1N} is the Nth root of the Love wave dispersion equation

$$\tan\{k_2^2 - K^2 H\} = \nu \frac{(K^2 - k_1^2)^{1/2}}{(k_2^2 - K^2)^{1/2}}\ \nu = \frac{\mu_1}{\mu_2} \quad \ldots(3)$$

corresponding to the layer thickness H. Moreover, $K_{1N} = \frac{\omega}{C_{1N}}$ $k = \omega/c$, where C_{1N} is the phase velocity of the Love waves of the Nth mode.

Let the total displacement field due to the presence of perfectly weak screens be written as

$$V_j^{\text{total}} = V^{(inc)} + V_j,\ (j = 1, 2, 3, 4) \quad \ldots(4)$$

where V_j represents the diffracted displacements in the jth region as shown in Fig. 1. The geometry of the problem leads to the following boundary conditions :

(a) At $z = 0, \infty > x > -\infty$ ...(5a)

$$V_1 = V_2$$

$$\frac{\partial V_1}{\partial Z} = \frac{\partial V_2}{\partial Z} .$$

(b) For $x < 0$.

$$\left.\begin{array}{l} \text{At } Z = h_1 + 0, \dfrac{\partial V_2}{\partial Z} \\ \text{At } Z = -h_1 - 0, \dfrac{\partial V_3}{\partial Z} \\ \quad Z = -h_2 + 0, \\ \text{At } Z = -h_2 - 0, \dfrac{\partial V_4}{\partial Z} \end{array}\right\} = -\frac{\partial V_2^{inc}}{\partial Z} = A\sigma_{2N} \sin(\sigma_{2N}\,\delta_i) \exp - iK_{1N}(x - x0) \quad ...(5b)$$

$$\delta_i = H - h_i : i = 1, 2.$$

(c) At $z = -H, -\infty < x < \infty$

$$\frac{\partial V_4}{\partial Z} = 0 \quad ...(5c)$$

(d) At $z = -h_1, x > 0$

$$V_2 = V_3, \frac{\partial V_2}{\partial Z} = \frac{\partial V_3}{\partial Z} \quad ...(5d)$$

(e) At $z = -h_2, x > 0$

$$V_3 = V_4, \frac{\partial V_3}{\partial Z} = \frac{\partial V_4}{\partial Z}. \quad ...(5e)$$

The displacements V_j ($j = 1; 2, 3, 4$) satisfy the differential equations

$$\frac{\partial^2 V_j}{\partial x^2} + \frac{\partial^2 V_j}{\partial Z^2} + k_{1,2}^2 V_j = 0 \; (j = 1; j = 2, 3, 4) \quad ...(6,7)$$

where $k_i = \dfrac{\omega}{\beta i}$, $| k_1 | < | k_2 |$. The differential equations (6) and (7) together with the boundary conditions (5a) through (5e) constitute the boundary value problem.

With a little effort the above mixed boundary value problem can be written, in the Fourier transformed plane, in terms of the following two Wiener-Hopf equations:

$$\frac{1}{\sigma_2^2} \frac{\sigma_n h}{\sinh \sigma_2 h} \frac{I(\alpha, h_2)}{I(\alpha, h_1)} \left\{ V'_{2+}(\alpha_1 - h_1) + V'_2 - (\alpha, h_1) \right\} - \frac{1}{\sigma_2^2} \frac{\sigma_n h}{\sinh \sigma_2 h}$$

(*equation continued on p. 589*)

$$\left\{V'_{4+}(\alpha, -h_2) + V'_{4-}(\alpha, -h_2)\right\} = h\,\{V_{3-}(\alpha, -h_1) - V_{2-}(\alpha, -h_1)\} \qquad \ldots(8)$$

$$\frac{1}{\sigma_2^2}\,\frac{\sigma_2 h}{\sinh \sigma_2 h}\left\{V'_{2+}(\alpha, -h_1) + V'_{2-} + (\alpha, h_1)\right\}$$

$$- \frac{1}{\sigma_2^2}\,\frac{\sigma_2 h}{\sinh \sigma_2 h}\,\frac{\sinh \sigma_2\,\delta_1}{\sigma_2\,\delta_1}\,\frac{\delta_1'}{\delta_2}$$

$$\times \left\{V'_{4+}(\alpha, -h_2) + V'_{4-}(\alpha, -h_2)\right\} = \{V_{3-}(\alpha, -h_2) - V_{4-}(\alpha, h_2)\} \qquad \ldots(9)$$

where

$$\sigma_i^2 = (\alpha^2 - k_i^2)^{1/2};\; I(\alpha, h_l) = \sigma_2 \sin \sigma_2\, h_l + \nu\sigma_1 \cosh \sigma_2 h_l$$

$$h = h_2 - h_1,\; \delta_l\; H - h_l.$$

In equations (8) and (9) the functions with subscript '+' are analytic in the domain $\mathrm{Im}\,(\alpha + k_1) > 0$ and those with subscript '−' are analytic in the domain $\mathrm{Im}\,(\alpha - k_1) < 0$.

3. Determination of the Wiener-Hopf Solution

The solution of the Wiener-Hopf equations (8) and (9) can be obtained by the usual procedure outlined by Noble[6]. Omitting the details of calculation, we can write the diffracted field in various regions as :

$$V_2(\alpha, Z) = -\frac{\sigma_2 \cosh \sigma_2 Z - \nu\sigma_1 \sinh \sigma_2 Z}{\sigma_2 \sinh \sigma_2 h_1 + \nu\sigma_1 \cosh \sigma_2 h_1}\,\frac{H_+(\alpha)}{T_+(\alpha)}\left(\frac{\alpha + k}{\alpha - k}\right)^{1/2}$$

$$\left[N_+(\alpha) - \frac{iA}{\sqrt{2\pi}}\,\frac{\sigma_2 N}{(\alpha - K_{1N})}\exp(iK_{1N}\,x0)\left\{\frac{T_+(K_{1N})\,\sin \sigma_{2N}\,\delta_1}{(K_{1N}+k_2)\,H_+(K_{1N})}\right.\right.$$

$$\left.\left. - \sin \sigma_{2N}\,\delta_2\,[P_+(\alpha) - P_+(K_{1N})]\right\}\right]. \qquad \ldots(10)$$

$$V_3(\alpha, Z) = \frac{\cosh \sigma_2 (Z + H)}{\sinh \sigma_2 H}\,\frac{H_+(\alpha)}{T_+(\alpha)}\left(\frac{\alpha + k_2}{\alpha - k_2}\right)^{1/2}$$

$$\left[N_+(\alpha) - \frac{iA}{2\pi}\,\frac{\sigma_{2N}\exp(ik_{1N}\,x0)}{(\alpha - K_{1N})} \times \frac{T_+(K_{1N})\sin(\sigma_{2N})\,\delta_1}{(K_{1N} + k_2)\,H_+(K_{1N})}\right.$$

$$\left. - \sin(\sigma_{2N}\,\delta_2)\,[P_+(\alpha) - P_+(K_{1N})]\right\}\Bigg]$$

$$- \frac{\cosh \sigma_2 (Z+h_1)\,H_+(\alpha)}{\sinh \sigma_2 h\; Y_+(\alpha)}\left(\frac{\alpha + k_2}{\alpha - k_2}\right)^{1/2}$$

(equation continued on p. 590)

$$\left[O_+(\alpha) - \frac{iA}{2\pi} \cdot \frac{\sigma_{2N} \exp(iK_{1N} x0)}{(\alpha - K_{1N})} \times \frac{Y_+(K_{1N}) \sin \sigma_{2N} \delta_2}{(K_{1N} + k_2) H_+(K_{1N})} \right.$$

$$\left. - \sin(\sigma_{2N} \delta_1) [W_+(\alpha) - W_+(K_{1N})] \Big\} \right]. \quad \ldots(11)$$

$$V_4(\alpha, Z) = \frac{\cosh \sigma_2 (Z + H)}{\sinh \sigma_2 \delta_2} \frac{H_+(\alpha)}{Y_+(\alpha)} \left(\frac{\alpha + k}{\alpha - k} \right)^{1/2}$$

$$\left[O_+(\alpha) - \frac{iA}{2\pi} \frac{\sigma_{2N} \exp(iK_{1N} x0)}{(\alpha - K_{1N})} \times \left\{ \frac{Y_+(K_{1N}) \sin(\sigma_{2N} \delta_1)}{(K_{1N} + k_2) H_+(K_{1N})} \right. \right.$$

$$\left. \left. - \sin(\sigma_{2N} \delta_1) [W_+(\alpha) - W_+(K_{1N})] \right\} \right]. \quad \ldots(12)$$

In obtaining these results, we have used the factorization

$$\frac{\sinh \sigma_2 h}{\sigma_2 h} = H_+(\alpha). H_-(\alpha)$$

$$\frac{I(\alpha, h_2)}{I(\alpha, h_1)} = T_+(\alpha). T_-(\alpha) \quad \ldots(13)$$

$$\frac{\sinh(\sigma_2 \delta_1)}{\sigma_2 \delta_1} \frac{\sigma_2 \delta_2}{\sinh \sigma_2 \delta_2} = Y_+(\alpha). Y_-(\alpha)$$

which have been explicitly obtained in the Appendix (A). Also, the splitting technique of Noble[6] is used to write the following additive decompositions (the explicity forms are given in Appendix B).

$$\frac{V'_{4+}(\alpha, -h_2)}{(\alpha + k_2) H_+(\alpha) T_-(\alpha)} = N_+(\alpha) + N_-(\alpha),$$

$$\frac{1}{(\alpha + k_2) H_+(\alpha) T_-(\alpha)} = P_+(\alpha) + P_-(\alpha) \quad \ldots(14)$$

$$\frac{V'_{2+}(\alpha, h_1)}{(\alpha + k_2) H_+(\alpha) Y_-(\alpha)} = O_+(\alpha) + O_-(\alpha)$$

$$\frac{1}{(\alpha + k_2) H_+(\alpha) Y_-(\alpha)} = W_+(\alpha) + W_-(\alpha).$$

4. The Transmitted Waves

We determine the transmitted waves in the three regions that are formed by the half planes in the layer. This can be done by taking inverse transforms. We do this for each region separately.

(*a*) *The region* $-h_1 < z < 0;\ x < 0$

The Fourier inversion formula gives

$$v_2(x, z) = \frac{1}{2\pi} \int_{c-\infty}^{c+\infty} \exp(-i\alpha x) V_2(\alpha, z) \, d\alpha \qquad \ldots(15)$$

where $V_2(\alpha, z)$ is given (10) and $\mathrm{Im}(k_1) > c > -\mathrm{Im}(k_1)$. For $x < 0$; $-h_1 < z < 0$, we can close the contour in the upper half plane. Lapwood[5] showed that the contributions to the surface waves come through the poles. The branch point contributions give rise to body waves which are of no interest to us for the present study. The integrand in eqn. (15) has simple poles at $\alpha = K_{1N}$ and at the zeros of $I(\alpha, h_1)$ located in the upper half plane. Using the relations (2) and (3), the contribution $v_{2,1}$ from the pole at $\alpha = K_{1N}$ can be written as

$$v_{2,1}(x, z) = -A \exp\{-iK_{1N}(x - x0\} \cos\{\sigma_{2N}(Z + H)\} \qquad \ldots(16)$$

which cancels exactly the incident Love wave as is to be expected. Let K_{2m} $(m = 1, 2, 3 \ldots)$ denote the zeros of $I(\alpha, h_1)$ in the upper half plane. Then using

$$\sigma_{1m} = (K_{2m}^2 - k_1^2)^{1/2}, \quad \sigma_{2m} = (k_2^2 - K_{2m}^2)^{1/2}$$

we can write

$$v_{2,2} = 2\pi i \sum_{m=1}^{\infty} \frac{\cos \sigma_{2m}(Z + H) K_{2m}}{\sigma_{2m} h_1 (\sin \sigma_{2m} h_1)} \cos(\sigma_{2m} h_1) \left(\frac{C_{2m}}{U_{2m}} - 1\right) \exp(-iK_{2m} x)$$

$$\left[\frac{K_{2m} + k_2}{K_{2m} - k_2}\right]^{1/2} \frac{H_+(K_{2m})}{T_+(K_{2m})} \left[N_+(K_{2m}) - \frac{iA}{2\pi} \frac{\sigma_{2N} \exp(iK_{1N} x0)}{(K_{2m} - K_{1N})}\right.$$

$$\left. \times \left\{\frac{T_+(K_{1N}) \sin \sigma_{2N} \delta_1}{H_+(K_{1N})(K_{1N} + k_2)} - \sin \sigma_{2N} \delta_2 [P_+(K_{2m}) - P_+(K_{1N})]\right\}\right]. \qquad \ldots(17)$$

In (17), we have used

$$\left.\frac{dI(\alpha, h_1)}{d\alpha}\right|_{\xi = K_{2m}} = \frac{\sigma_{2m}^2 h_1}{K_{2m} \cos(\sigma_{2m} h_1) \left(\frac{C_{2m}}{U_{2m}} - 1\right)} \qquad \ldots(18)$$

where $c_{2m} = \frac{\omega}{K_{2m}}$ is the phase velocity of the Love type waves of the mth mode. It may be noted that (17) requires the relations $I(K_2 m) = 0$ which is equivalent to

$$\tan \sigma_{2m} h_1 = \nu \frac{\sigma_{1m}}{\sigma_{2m}} . \qquad \ldots(19)$$

Since (19) is the relation for the propagation Love type waves in layered structure consisting of a semi-infinite solid of rigidity μ_1 covered by a surface layer of uniform thickness h_1 and rigidity μ_2, $\nu_{2,2}$ is therefore a Love propagating in the geometry as show in Fig. 1.

(*b*) *The region* $-h < z < -h; z < 0$

The transmitted wave $v_3(x, z)$ in this region is determined by applying inversion formula to eqn. (11). The contour of integration is closed in the upper half plane. The integrand has simple poles at $\alpha = K_{1N}$ and at the zeros of sinh $\sigma_2 h$ lying in the upper half plane. The contribution $v_{3,1}(x, z)$ arising from the pole at $\alpha = K_{1N}$ cancels the incident wave v_2^{inc} in this region. The contribution from the poles at $\alpha = ip_n$, where $P_n = \left[\frac{n^2\pi^2}{h^2} - k^2\right]^{1/2}$, denoted by $v_{3,2}$ is given by

$$v_{3,2} = \sqrt{2\pi i}\sum_{n=0}^{\infty} \exp(p_n x)\left[\frac{ip_n + k_2}{ip_n - k_2}\right]^{1/2} H_+(ip_n)\left[\frac{{}^*\cos\frac{\pi n}{h}(Z+H)}{T_+(ip_n)}\right.$$

$$- \frac{iA}{2\pi}\frac{\sigma_{2N}\exp(iK_{1N}x0)}{ip_n - K_{1N}}\left\{\frac{T_+(K_{1N})\sin(\sigma_{2N}\delta_1)}{(K_{1N}+k_2)H_+(K_{1N})}\right.$$

$$\left. - \sin\sigma_{2N}\delta_2\left[(P_+(ip_n) - P_+(K_{1N})\right]\right\}$$

$$- \frac{\cos\frac{n\pi}{h}(Z+h_1)}{Y_+(ip_n)}\left[O_+ - \frac{iA}{\sqrt{2\pi}}\frac{\sigma_{2N}\exp(iK_{1N}x0)}{(i_{pn} - K_{1N})}\right.$$

$$\left.\left.\times\left\{\frac{Y_+(K_{1N})\sin(\sigma_{2N}\delta_2)}{(K_{1N}+k_2)H_+(K_{1N})} - \sin(\sigma_{2N}\delta_1)(W_+(iP_n) - W_+(K_{1N}))\right\}\right]\right]^* \quad \ldots(20)$$

Note that in deriving eqn. (20), he have used the relation sinh $\sigma_2 h = 0$, which is the dispersion relation for waves in an infinite strip of uniform thickness δ and rigidity μ_2 with weak upper and lower surfaces at $z = -h_2$ and $z = -h_1$ respectively.

(*c*) *The region* $-H < z < -h_2; x < 0$

To find the transmitted waves in this range, we apply inversion formula to eqn. (12). Closing the contour of integration in the upper half plane, we find that the integrand has simple poles at $\alpha = K_{IN}$ and at zeros of sinh $\sigma_2\delta_2$ that lie in the upper half plane. The contribution, $v_{4,1}$, from $\alpha = K_{IN}$ exactly cancels the incident wave v_2^{inc} in this region. The poles arising from the zeros of sinh $\sigma_2\delta_2$ are $\alpha = ip'_n = \left\{k_2^2 - \frac{n^2\pi^2}{\delta^2}\right\}^{1/2}$, $n = 1, 2, 3, \ldots$ These poles give rise to the contribution

$$V_{4,2} = \sqrt{2\pi i}\sum_{n=0}^{\infty} \exp(p'_n x)\cos\frac{n\pi}{\delta_2}(Z+H)\left[\frac{ip'_n + k_2}{ip'_n - k_2}\right]^{1/2}\frac{H_+(ip'_n)}{Y_+(ip'_n)}$$

(*equation continued on p.* 593)

$$\times \left[Q_+ \; (ip'_n) - \frac{iA}{\sqrt{2\pi}} \frac{\sigma_{2N} \exp (iK_{IN} \, x_0)}{(ip'_n - K_{1N})} \left\{ \frac{Y_+ (K_{1N}) \sin \sigma_{2N} \, \delta_2}{(K_{1N} + k_2) \, H_+ \, (K_{1N})} \right. \right.$$

$$\left. \left. - \sin \sigma_{2N} \, \delta_1 \, (W_+ \; (ip'_n) \, W_+ \, (K_{1N})) \right\} \right] . \qquad ...(21)$$

The dispersion equation satisfied by $v_{4,2}$ corresponds to the relation $\sinh \sigma_2 \, \delta_2 = 0$, that is $\alpha = \left[k_2^2 - \frac{n^2 \, \pi^2}{\delta^2} \right]^{1/2}$, which is the dispersion relation for waves in an infinite strip of uniform thickness δ_2 and rigidity μ_2 with free upper surface and free/weak lower surface respectively.

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APPENDIX A

The factorization of the function, involved in eqn. (8) and (9), has been fully described by Sato[7]. We only quote the results here.

(a) Let us write

$$\frac{\sinh \sigma_2 \, \delta_2}{\sigma_2 \, \delta} = \prod_{n=1}^{\infty} \left\{ \bar{p}_n^2 \, \bar{\delta}_n^2 + \alpha^2 \, \bar{\delta}_n^2 \right\} = \bar{H} (\alpha) \qquad ...(A.1)$$

where

$$\bar{p}_n \, \bar{\delta}_n = \left[1 - k_2^2 \, \bar{\delta}_n^2 \right]^{1/2} - i \, (k_2^2 \, \bar{\delta}_n^2 - 2), \; \bar{\delta}_n = \frac{\delta_n}{\pi n}. \qquad ...(A.2)$$

Also $\bar{H} (\alpha) = \bar{H}_+ \, . \, \bar{H}_- \, (\alpha)$, where

$$H_{\pm}(\alpha) = \prod_{n=1}^{\infty} \bar{p}_n \bar{\delta}_n \mp i\alpha \; \bar{\delta}_n + \exp\{\mp i\alpha \bar{\delta} + \bar{\chi}(\alpha)\}. \qquad \text{...(A.3)}$$

If

$$\bar{\chi}(\alpha) = -i\alpha \; \frac{\delta}{\pi}\left[1 - c - \log\frac{\alpha\delta}{\pi} + \frac{\alpha\delta}{2}\right], \text{ then } \bar{H}_{\pm}(\alpha) \sim |\alpha|^{-1/2}$$

as $|\alpha| \to \infty$

in appropriate half planes. Hence

$$\frac{\sinh \sigma_2 \delta}{\sigma_2 \delta} = \bar{H}_+(\alpha) \, . \, \bar{H}_-(\alpha). \qquad \text{...(A.4)}$$

(b) If $\pm \hat{K}_{1m}$ and $\pm \hat{K}_{2m}$ $(m = 1, 2, ...)$ denote the zeros of $I(\alpha, h_2)$ and $I(\alpha, h_1)$ respectively, we can write

$$\frac{I(\alpha, h_3)}{I(\alpha, h_1)} = \prod_{m=1}^{\infty} \left(\frac{\alpha^2 - \hat{K}^2_{1m}}{2 - K^2_{2m}}\right) \frac{G_1(\alpha)}{G_2(\alpha)}$$

where

$$G_1(\alpha) = I(\alpha, h_2) / \prod_{m=1} (\alpha^2 - \hat{K}^2_{1m})$$

$$G_2(\alpha) = I(\alpha, h_1) / \prod_{m=1}^{\infty} (\alpha^2 - \hat{K}^2_{2m})$$

and $G_{1,2}(\alpha)$ has no zeros.

Let

$$Q(\alpha) = \frac{G_1(\alpha)}{G_2(\alpha)} = Q_+(\alpha) \, . \, Q_-(\alpha)$$

then

$$\log Q_+(\alpha) = \frac{1}{2\pi i} \int_{\Gamma} \log \frac{Q_+(\omega)}{(\omega)\,\alpha} \, d\omega; \; Q_-(\alpha) = Q_+(-\alpha),$$

$$\frac{I(\alpha, h_2)}{I(\alpha, h_1)} = \prod_{m=1}^{\infty} \frac{(\alpha^2 - \hat{K}^2_{1m})}{(\alpha^2 - \hat{K}^2_{2m})} Q_+(\alpha) \, . \, Q_-(\alpha) = T_+(\alpha) \, T_-(\alpha) \qquad \text{...(A.5)}$$

where

$$T_{\pm}(\alpha) = Q_{\pm}(\alpha) \prod_{m=1}^{\infty} \left(\frac{\alpha + \hat{K}_{1m}}{\alpha + \hat{K}_{2m}}\right). \qquad \text{...(A.6)}$$

(c) The parallel calculation from (A1) to (A4) for $\frac{\sinh \sigma_2 \delta_1}{\sigma_2 \delta_1}$ and $\frac{\sinh \sigma_2 \delta_2}{\alpha_2 \delta_2}$ lead to

$$\frac{\sinh \alpha_2 \delta_1}{\sigma_2 \delta_1} \cdot \frac{\sigma_2 \delta_2}{\sinh \sigma_2 \delta_2} = Y_+ (\alpha) \, . \, Y_- (\alpha) \qquad \ldots(A.7)$$

where

$$\frac{\sinh \sigma_2 \delta_1}{\sigma_2 \delta_1} = Y_{1+} (\alpha) \, . \, Y_{1-} (\alpha)$$

$$\frac{\sinh \sigma_2 \delta_2}{\sigma_2 \delta_2} = Y_{2+} (\alpha) \, . \, Y_{2-} (\alpha)$$

and

$$Y_\pm (\alpha) = \frac{Y_1 \pm (\alpha)}{Y_2 \pm (\alpha)} .$$

Appendix B

Following the general decomposition theorem (Noble[6]), the explicit respresentations $N_\pm (\alpha)$, $P_\pm (\alpha)$, $O_\pm (\alpha)$ and $W_\pm (\alpha)$ are given by

$$N_\pm (\alpha) = \pm \frac{1}{2\pi i} \int_{\substack{ic+\infty \\ id+\infty}} \frac{V'_{4+} (\xi, -h_2) \, d\xi}{(\xi + k_2) H_+ (\xi) T_- (\xi) (\xi - \alpha)} \qquad \ldots(B.1)$$

$$P_\pm (\alpha) = \pm \frac{1}{2\pi i} \int^{\substack{ic-\infty \\ id-\infty}} \frac{d\xi}{(\xi + k_2) H_+ (\xi) T_-(\xi) (\xi - \alpha)} \qquad \ldots(B.2)$$

$$O_\pm (\alpha) = \pm \frac{1}{2\pi i} \int \frac{V'_{2+} (\xi, -h) \, d\xi}{(\xi + k_2) H_+ (\xi) Y_- (\xi) (\xi - \alpha)} \qquad \ldots(B.3)$$

$$W_\pm (\alpha) = \pm \frac{1}{2\pi i} \int \frac{d\xi}{(\xi + k_2) H_+ (\xi) Y_- (\xi) (\xi - \alpha)} \qquad \ldots(B.4)$$

where $-\mathrm{Im}\,(k_1) < c < \mathrm{Im}\,(\alpha) < d < \mathrm{Im}\,(k_1)$.

The integrals in (B1), (B2), (B3) and (B4) can be calculated by the contour integration method. Let us consider

$$N_+ (\alpha) = \frac{1}{2\pi i} \int_{ic-\infty}^{ic+\infty} \frac{V'_{4+} (\xi) \, d\xi}{(\xi + k_2) H_+ (\xi) (\xi - \alpha) T_- (\xi)} \cdot \qquad \ldots(B.5)$$

For α lying in $(-\eta, \eta)$ we can write $\alpha = -\eta \cos \theta$, where $|\theta| < \pi$. Thus, eqn. (B5) can be written as

$$N_+(-\eta\cos\theta) = \frac{1}{2\pi i}\int_{ic-\infty}^{ic+\infty} \frac{V'_{4+}(\xi_1 - h_2)\prod_{m=1}^{\infty}(\xi - \hat{K}_{2m})\,d\xi}{(\xi+k_2)\,H_+(\xi)\,T_-(\xi)\,(\xi+\eta\cos\theta)} \qquad \ldots(B.6)$$

where $T_-(\xi) = Q_-(\xi)\prod_{m=1}^{\infty}\left[\frac{\xi - \hat{K}_{1m}}{\xi - \hat{K}_{2m}}\right]$, and $Q_-(\xi)$ has no zeros. Putting the value of $T_-(\xi)$ in (B6), we obtain

$$N_+(-\eta\cos\theta) = \frac{1}{2\pi i}\int_{ic-\infty}^{ic+\infty} \frac{V'_{4+}(\xi_1 - h_2)\prod_{m=1}(\xi - \hat{K}_{2m})\,d\xi}{(\xi+k_2)\,H_+(\xi)\,Q_-(\xi)\prod_{m=1}^{\infty}(\xi - \hat{K}_{1m})\,(\xi+\eta\cos\theta)}. \qquad \ldots(B.7)$$

Closing the line of integration by a semi-circle in the upper half plane the poles captured are $\xi = -\eta\cos\theta$, $\xi = \hat{K}_{1m}$ $(m = 1, 2\ldots)$. Thus

$$N_+(-\eta\cos\theta) = \sum_{m=1}^{\infty} \frac{V'_{4+}(K_{1l} - h_2)}{(\hat{K}_{1l} + k_2)\,H_+(\hat{K}_{1l})\,Q_+(\hat{K}_{1l})}\;\frac{\prod_{m=1}^{\infty}(\hat{K}_{1l} - \hat{K}_{2m})}{\prod_{m=1}^{\infty}(\hat{K}_{1l} - \hat{K}_{1m})}$$

$$+\frac{V'_{4+}(-\eta\cos\theta, -h_2)}{(K_2 - \eta\cos\theta)\,H_+(-\eta\cos\theta)\,T_-(-\eta\cos\theta)}. \qquad \ldots(B.8)$$

The other integrals can be evaluated similarly.