CHARACTERIZATIONS OF $T_{1/2}$-SPACES USING GENERALIZED $V$-SETS

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In this note the authors investigate properties of generalized $v$-sets in a
topological space and characterize the notion of $T_{1/2}$-spaces due to Levine$^3$
by using generalized $v$-sets. Finally, a new class of topological spaces is
introduced.

1. INTRODUCTION

Levine$^3$ introduced the concept of generalized closed sets of a topological space
and a class of topological spaces called $T_{1/2}$-spaces. The author$^3$ proves every $T_1$-space
is $T_{1/2}$ and every $T_{1/2}$-space is $T_0$, although neither implication is reversible. Dunham$^2$
defined a new closure operator by using the generalized closed set and investigated a
new topology and its properties. And, the class of $T_{1/2}$-spaces is characterized by the
new topology (cf. Theorem 3.7 of Dunham$^2$). In this note we characterize the class of
$T_{1/2}$-spaces by using generalized $v$-set. And we introduce a new class of topological
spaces named $T^v$-spaces.

2. PRELIMINARIES

Throughout this note, $(X, \tau)$ denotes a topological space with a topology $\tau$ on
which no separation axioms are assumed unless explicitly stated. We recall some
definitions and properties of Maki$^4$. The most of them can be quoted in the sequel.

Definition 2.1—In a topological space $(X, \tau)$, a subset $B$ is a $v$-set (resp. $A$-set),
if $B = B^v$ (resp. $B = B^A$), where $B^v = \bigcup \{F : F \subset B, X - F \in \tau\}$ and $B^A = \bigcap \{U : U \supset B, U \in \tau\}$.\footnote{This paper is dedicated to Professor Hiroshi Toda on his 60th birthday.}
Then, in (2.5), (2.7) and Remark 2.8 of Maki\textsuperscript{4}, we have

\[(2.2) \ (X - B)^A = X - B^r \text{ for every subset } B.\]

(2.3) Let \( \{ B_j : j \in J \} \) be a family of subsets of \( X \).

Then

\[ (\cup \{ B_j : j \in J \})^r \supset \cup \{ B_j^r : j \in J \} \text{ and } (\cup \{ B_j : j \in J \})^A = \cup \{ B_j^A : j \in J \}. \]

Moreover we have

\[(2.4) \ B^r \subset B \subset \text{cl } (B), \text{ where cl } (B) \text{ is the closure of } B \text{ in } (X, \tau). \]

\textit{Definition 2.5}—In a topological space \((X, \tau)\), a subset \( B \) is a generalized \( \Delta \)-set\textsuperscript{4} (abbreviated by \( g. \Delta \)-set) if \( B^A \subset F \) whenever \( B \subset F \) and \( F \) is closed. A subset \( B \) is a generalized \( v \)-set\textsuperscript{4} (abbreviated by \( g. v \)-set) of \((X, \tau)\) if \( X - B \) is a \( g. \Delta \)-set of \((X, \tau)\).

\textit{Definition 2.6}—For a subset \( B \) of a topological space \((X, \tau)\), we define the following subsets:

\[ c^A (B) = \cap \{ U : \overline{B \cup U}, \ U \in \mathcal{D}^A \} \text{ and } \text{int}^r (B) = \cup \{ F : \overline{B \supset F}, \ F \in \mathcal{D}^r \}, \]

where \( \mathcal{D}^A \) (resp. \( \mathcal{D}^r \)) denotes the set of all \( g. \Delta \)-sets (resp. \( g. v \)-sets) in \((X, \tau)\).

In Proposition 3.2, Example 3.3, Proposition 3.5 and Theorem 4.6 of Maki\textsuperscript{4} we have the followings.

(2.7) Every \( \Delta \)-set (resp. \( v \)-set) is a \( g. \Delta \)-set (resp. \( g. v \)-set).

(2.8) Let \( J \) be an indexed set. If \( B_j \in \mathcal{D}^A \) (resp. \( B_j \in \mathcal{D}^r \)) for all \( j \in J \), then \( \cup \{ B_j : j \in J \} \in \mathcal{D}^A \) (resp. \( \cap \{ B_j : j \in J \} \in \mathcal{D}^r \}).

The intersection of two \( g. \Delta \)-sets is not a \( g. \Delta \)-set in general.

(2.9) For each \( x \in X \), \( \{ x \} \) is an open set or a \( g. v \)-set.

(2.10) \( c^A \) is a Kuratowski closure operator on \( X \).

\textit{Definition 2.11} (cf. 4.2) Definition 4.7 and Proposition 4.8 (i) of Maki\textsuperscript{4})—Let \( \tau^A \) be the topology on \( X \) generated by \( c^A \) in the usual manner, i. e., \( \tau^A = \{ B : B \subset X, \ c^A (X - B) = X - B \} \). Then \( \tau^A = \{ B : B \subset X, \ \text{int}^r (B) = B \} \) since \( c^A (X - B) = X - \text{int}^r (B) \) for every subset \( B \) of \((X, \tau)\).

In Theorem 4.8 (iii) of Maki\textsuperscript{4} we have

(2.12) \( \mathcal{F} \subset \tilde{\tau} \subset \mathcal{D}^r \subset \tau^A \), where \( \tilde{\tau} \) (resp. \( \mathcal{F} \)) denotes the family of all \( v \)-sets (resp. closed sets) in \((X, \tau)\).

3. \textbf{Generalized \( v \)-sets}

In this section we characterize the notion of generalized \( v \)-sets of Definition 2.5 by using \( v \)-operations and we obtain results concerning such sets.
Proposition 3.1—A subset $B$ of a topological space $(X, \tau)$ is a g. $v$-set if and only if $U \subseteq B^v$ whenever $U \subseteq B$ and $U$ is open.

**Proof : Necessity**—Let $U$ be an open subset of $(X, \tau)$ such that $U \subseteq B$. Then, since $X - U$ is closed and $X - U \supset X - B$, we have $X - U \supset X - B^v$ according to (2.2) and Definition 2.5.

**Sufficiency**—Let $F$ be a closed subset of $(X, \tau)$ such that $X - B \subseteq F$. Since $X - F$ is open and $X - F \subseteq B$, by assumption we have $X - F \subseteq B^v$. Then, $F \supset (X - B)^\Lambda$ by (2.2), and $X - B$ is a g.$\Lambda$-set (i. e., $B$ is a g.$v$-set).

Proposition 3.2—Let $B$ be a g.$v$-set in a topological space $(X, \tau)$. Then, for every closed set $F$ such that

$$ B^v \cup (X - B) \subseteq F, \quad F = X \text{ holds.} $$

**Proof :** The assumption $B^v \cup (X - B) \subseteq F$ implies $(X - B^v) \cap B \supset X - F$. Since $B$ is a g. $v$-set, by Proposition 3.1 we have $B^v \supset X - F$ and hence $\phi = (X - B^v) \cap B^v \supset X - F$. Therefore we have $X = F$.

Corollary 3.3—Let $B$ be a g. $v$-set of $(X, \tau)$. $B^v \cup (X - B)$ is a closed set if and only if $B$ is a $v$-set.

**Proof :** The proof of necessity is obtained by Proposition 3.2. The converse is obvious.

Remark 3.4: We give an example of a topological space which shows that the condition of Proposition 3.2 (i. e., $B$ is a g. $v$-set) cannot be removed. Let $X$ be the set of all real numbers. On $X$, we can define the open sets of a topology to be $\phi$ and any subset of $X$ that contains a particular point $p$, which is called a particular point topology, say $\tau$. Let $B = \{x \in X : x \leq p - 1\} \cup \{p\}$, then $B^v = B - \{p\}$. In $(X, \tau)$, $B$ is not a g. $v$-set, and $B^v \cup (X - B) = X - \{p\}$ holds and it is a closed set.

Proposition 3.5—Let $B$ be a subset of topological space $(X, \tau)$ such that $B^v$ is closed. If $X = F$ holds for every closed subset $F$ such that

$$ F \supset B^v \cup (X - B), \quad \text{then} \quad B \text{ is a g.}$v$-set.

**Proof :** Let $U$ be an open subset contained in $B$. According to assumption, $B^v \cup (X - U)$ is closed and it contains $B^v \cup (X - B)$. It follows that $B^v \cup (X - U) = X$ and hence $B^v \supset U$. Then $B$ is a g. $v$-set.

Remark 3.6: Let $X$ be a finite set, and let $\tau$ be a topology on $X$. In the topological space $(X, \tau)$, $B^v$ is closed for every subset $B$ of $X$.

Remark 3.7: The following example shows that the condition of Proposition 3.5 (i. e., $B^v$ is a closed set) cannot be removed. Let $X$ be the set $[-1, 1] = \{x \in R : -1 \leq x \leq 1\}$, and let $\tau$ be the overlapping interval topology, which is generated from sets of the form $[-1, b)$ for $b > 0$ and $(a, 1]$ for $a < 0$. Let $B = [-1, 1/2]$. Then
$B^\tau = \{-1, 0\}$ is not closed. Any closed set containing $B^\tau \cup \{X - B\}$ is $X$, however $B$ is not a $g. v$-set.

4. Characterizations of $T_{1/2}$-Spaces

In a topological space $(X, \tau)$ a subset $B$ of $X$ is said to be $g.$ closed\textsuperscript{3} if $\text{cl}(B) \subset U$ whenever $B \subset U$ and $U$ is open.

Definition 4.1\textsuperscript{3}—A topological space $(X, \tau)$ is said to be a $T_{1/2}$-space, if every $g.$ closed set is closed.

The notion of $g. v$-sets in a topological space $(X, \tau)$ does not coincide with the one of $g.$ closed sets even if $(X, \tau)$ is a finite topological space.(cf. Remark 4.10 of Maki\textsuperscript{4}). However the class of $T_{1/2}$-spaces is characterized by the notion of $g. v$-sets (cf.. Theorem 3.7 of Dunham\textsuperscript{2}). In the following theorem we give the characterization of the class of $T_{1/2}$-spaces.

Theorem 4.2—The statements about a topological space $(X, \tau)$ are equivalent.

(i) $(X, \tau)$ is a $T_{1/2}$-space.
(ii) Every $g. v$-set is a $v$-set.
(iii) Every $\tau^A$-open set is a $v$-set.

Proof: (i) $\rightarrow$ (ii) Supposed that there exists a $g. v$-set $B$ which is not a $v$-set. Since $B^\tau \not\subset B$ there exists a point $x$ of $B$ such that $x \not\in B^\tau$. Then the singleton $\{x\}$ is not closed. According to Theorem 2.2 of Dunham\textsuperscript{2} the complement of $\{x\}$ (i. e., $X - \{x\}$) is a $g.$ closed set. On the other hand, we have $\{x\}$ is not open since $B$ is a $g. v$-set and $x \not\in B^\tau$. Therefore we have $X - \{x\}$ is not closed but it is a $g.$ closed. This contradicts to the assumption that $(X, \tau)$ is a $T_{1/2}$-space.

(ii) $\rightarrow$ (i) Suppose that $(X, \tau)$ is not a $T_{1/2}$-space. Then, there exists a $g.$ closed set $B$ which is not closed. Since $B$ is not closed, there exists a point $x$ such that $x \in B$ and $x \in \text{cl}(B)$. In (2.9) we have the singleton $\{x\}$ is an open set or it is a $g. v$-set. When $\{x\}$ is open, we have $\{x\} \cap B \neq \phi$ because $x \in \text{cl}(B)$. This is a contradiction. Let us consider the case: $\{x\}$ is a $g. v$-set. If $\{x\}$ is not closed, we have $\{x\}^\tau = \phi$ and hence $\{x\}$ is not a $v$-set. This contradicts to (ii). Next, if $\{x\}$ is closed, we have $X - \{x\} \subset \text{cl}(B)$ (i. e., $x \in \text{cl}(B)$). In fact, the open set $X - \{x\}$ contains the set $B$ which is a $g.$ closed set. Then, this also contradicts to the fact that $x \in \text{cl}(B)$. Therefore $(X, \tau)$ is a $T_{1/2}$-space.

(ii) $\rightarrow$ (iii) Let $B$ be a $\tau^A$-open set, i. e., $B = \text{int}^v(B)$. We claim that $\text{int}^v(B)$ is $v$-set under given assumptions. Then $B$ is a $v$-set. To prove the claim we note that $\tau^\sim = \tau^v$ according to the assumption (ii) and (2.7). By using (2.3) and the fact that $\tau^v = \tau$ we have $(\text{int}^v(B))^\tau = (\bigcup \{F : B \supset F, F \in \tau^\sim\})^\tau \cup \bigcup \{F^* : B \supset F, F \in \tau\} = \text{int}^v(B)$. Then, by (2.4), we have $(\text{int}^v(B))^\tau = \text{int}^v(B)$, and hence $\text{int}^v(B)$ is a $v$-set.

(iii) $\rightarrow$ (ii) Let $B$ be a $g. v$-set. Then $\text{int}^v(B) = B$ by Definition 2.6. It follows from the fact in Definition 2.11 that $B \in \tau^A$. Hence $B$ is a $v$-set, by (iii).
Corollary 4.3 (Dunham\textsuperscript{2}, Theorem 2.6)—A topological space \((X, \tau)\) is \(T_{1/2}\) if and only if every singleton in \((X, \tau)\) is open or closed.

**Proof:** *Necessity*—Let \(x \in X\). If \(\{x\}\) is not open, then it is a \(g.v\)-set by (2.9). It follows from Theorem 4.2 and assumption that \(\{x\}\) is a \(v\)-set. Since \(\{y\}^v = \emptyset\) for every non-closed set \(\{y\}\), we have \(\{x\}\) is a closed set. Therefore \(\{x\}\) is open or closed.

*Sufficiency*—Suppose that \((X, \tau)\) is not a \(T_{1/2}\)-space. Then, there exists a \(g.v\)-set \(B\) which is not a \(v\)-set. Then there exists a point \(x \in X\) such that \(x \in B\) and \(x \not\in B^v\). If \(\{x\}\) is closed, \(B^v\) contains the closed set \(\{x\}\). This is impossible. If \(\{x\}\) is open, the closed set \(X - \{x\}\) contains \(B^v \cup (X - B)\). By using Proposition 3.2 we have \(X - \{x\} = X\) which is impossible. Therefore \((X, \tau)\) is a \(T_{1/2}\)-space.

Corollary 4.4—(i) For any topological space \((X, \tau), (X, \tau^A)\) is a \(T_{1/2}\)-space.

(ii) For any topology \(\tau, (\tau^A)^\sim = (\tau^A)^\sim\) where \((\tau^A)^\sim\) is the family of all \(v\)-sets in the topological space \((X, \tau^A)\).

**Proof:** (i) Let \(x \in X\). If \(\{x\}\) is open (i.e., \(\{x\} \in \tau\)), then \(X - \{x\} \in \tau^A\) (i.e., \(\{x\}\) is \(\tau^A\)-closed), by (2.12). If \(\{x\}\) is not open then \(\{x\}\) is a \(g.v\)-set by (2.9). Then by (2.12) we have \(\{x\} \in \tau^A\) (i.e., \(\{x\}\) is \(\tau^A\)-open). Therefore every singleton in \(X\) is open or closed in \((X, \tau^A)\). By Corollary 4.3 we obtain that \((X, \tau^A)\) is a \(T_{1/2}\)-space.

(ii) It follows from (i) and Theorem 4.2 that \((\tau^A)^\sim \subseteq (\tau^\tau)^\sim\). By using (2.12) we obtain the equality.

5. Comparisons

We define a class of topological spaces called \(T^v\)-spaces as an analogy to \(T_{1/2}\)-spaces, and we investigate relations between \(T^v\)-spaces and several spaces.

**Definition 5.1**—A topological space \((X, \tau)\) is said to be a \(T^v\)-space, if every \(\tau^A\)-open set is a \(g.v\)-set (i.e., \(D^v = \tau^A\)).

**Lemma 5.2**—For a topological space \((X, \tau)\), every singleton of \(X\) is \(g.A\)-set if and only if \(G = G^v\) holds for every open set \(G\).

**Proof:** *Necessity*—Let \(G\) be an open set. Let \(y \in X - G\), then \(\{y\}^A \subseteq X - G\) by assumption. By using (2.3) we have \(X - G \supseteq \bigcup \{\{y\}^A : y \in X - G\} = (X - G)^A\), and hence \(X - G = (X - G)^A\). Then it follows from (2.2) that \(G = G^v\).

*Sufficiency*—Let \(x \in X\) and \(F\) be a closed set such that \(\{x\} \subseteq F\). Since \(X - F = (X - F)^v = X - F^A\), we have \(F = F^A\). Therefore we have \(\{x\}^A \subseteq F^A = F\). Hence \(\{x\}\) is a \(g.A\)-set.

**Proposition 5.3**—For a topological space \((X, \tau)\), we have the following implications.

(i) If \((X, \tau)\) is a \(T_{1/2}\)-space then it is a \(T^v\)-space.
(ii) If \((X, \tau)\) is an \(R_0\)-space then it is a \(T^v\)-space.

**Proof**: (i) In (2.12) we have \(\sim \subset \mathcal{D}^v \subset \tau^A\). By using Theorem 4.2 we have \(\sim = \tau^A\). Therefore we obtain \(\tau^A = \mathcal{D}^v\). Hence, \((X, \tau)\) is a \(T^v\)-space.

(ii) To prove (ii) we recall the following fact. In Theorem 2.2 of Dubé\(^1\), it is shown that \((X, \tau)\) is an \(R_0\)-space if and only if any open set \(G\) in \((X, \tau)\) can be expressed as \(G = \bigcup \{F : F \subset G, X - F \in \tau\}\), i.e., \(G = G^v\) (cf. Definition 2.1). Let \(B\) be a subset of \((X, \tau)\). It follows from Lemma 5.2 and (2.8) that \(B = \bigcup \{\{b\} : b \in B\}\) is a \(g.\Lambda\)-set. Thus we have every subset of \((X, \tau)\) is a \(g.\nu\)-set, and hence \(\mathcal{D}^v\) coincides with the discrete topology of \(X\). By using (2.12) we obtain \(\mathcal{D}^v = \tau^A\). Hence, \((X, \tau)\) is a \(T^v\)-space.

The following examples show that none of these implications is reversible.

**Example 5.4**—Let \(X = \{a, b, c\}\) and let \(\tau = \{\phi, \{a\}, \{b, c\}, X\}\). The space \((X, \tau)\) is not a \(T^v\)-space. Since \(\{a\}^v = \{a, b\}^v = \{a, c\}^v = \{a\}, \{b\}^v = \{c\}^v = \phi\) and \(\{b, c\}^v = \{b, c\}\), we have \(\mathcal{D}^v = \tau^A\). These imply that \((X, \tau)\) is not \(T_{1/2}\) but \(T^v\).

**Example 5.5**—Let \(X = \{a, b, c\}\) and let \(\tau = \{\phi, \{a\}, \{a, b\}, X\}\). Then the space \((X, \tau)\) is not \(R_0\) but \(T^v\). In fact, we have \(\mathcal{D}^v = \{\phi, \{b\}, \{c\}, \{b, c\}, X\} = \tau^A\) since \(\{a\}^v = \{b\}^v = \{a, b\}^v = \phi\), \(\{a, c\}^v = \{c\}\) and \(\{b, c\}^v = \{b, c\}\). Thus, \((X, \tau)\) is not a \(T^v\)-space. Since \(\{a\}^v \neq \{a\}\) for an open set \(\{a\}\), \((X, \tau)\) is not an \(R_0\)-space.

\(T^v\) is independent of \(T_0\) an Example 5.4 and the following example show.

**Example 5.6**—Let \(X\) be the set of all real numbers \(R\) and let \(\tau = \{\phi, X\} \cup \{\{a, \infty\} : a \in X\}\), where \(\{a, \infty\} = \{x \in X : a < x\}\). Then the space \((X, \tau)\) is not \(T^v\) but \(T_0\). In fact, we obtain that \(\tau^A\) coincides with the discrete topology of \(R\) however \(\{a\}\) \(\cup \{a, \infty\}\) is not a \(g.\nu\)-set.

From the results in this action we obtain the following diagram:

\[
\begin{array}{ccc}
T_{1/2} & \longrightarrow & T^v \\
\downarrow & & \downarrow \\
T_0 & \longrightarrow & R_0 \\
\end{array}
\]

where \(A \rightarrow B\) (resp. \(A \rightarrow B\)) represents that \(A\) implies \(B\) (resp. \(A\) does not always imply \(B\)).

In the following proposition we have a further result concerning the transfer of properties from \((X, \tau)\) to \((X, \tau^A)\) (cf. Corollary 4.4 (i)).

**Proposition 5.7**—If \((X, \tau)\) is an \(R_0\)-space then \((X, \tau^A)\) is \(T_1\)-space.
PROOF: By using Theorem 2.2 of Dube' and Lemma 5.2 that every singleton \\{x\\} of X is a g. A-set. Then we have c^A (\\{x\\}) = \{x\\} and hence \{x\\} is a \tau^A-closed set. Therefore every singleton is closed in (X, \tau^A).

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