

## EIGENFUNCTION EXPANSION METHOD TO THERMOELASTIC AND MAGNETO-THERMOELASTIC PROBLEMS

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The fundamental equations of the general one dimensional problems of thermoelasticity and magneto-thermoelasticity have been written in the form of an inhomogeneous vector-matrix differential equation and solved in the Laplace transform domain by the eigen-function expansion method. It has been pointed out that a broad class of problems in mechanics can be solved by this approach. Applications to problems pertaining to an infinite medium due to instantaneous heat sources are presented and compared with existing literature. Finally, the solution for the space time domain has been made by numerical methods and the results are analysed.

### NOMENCLATURE

The nomenclature is as follows :

- H** = magnetic field
- B** = magnetic induction vector
- j** = current density vector
- E** = electric field vector
- u** = mechanical displacement vector
- H** =  $H_0 + h$ ,  $H_0 = [H_1, H_2, H_3]$ ,  $h = [h_x, h_y, h_z]$
- H<sub>0</sub>** = primary magnetic field
- h** = perturbation of **H**
- $\tau_{ij}$  = mechanical stress tensor
- $e_{ij}$  = strain tensor
- T** = differential temperature distribution
- $\mu_e$  = magnetic permeability
- $\nu_B = \frac{1}{\eta\mu_e}$  , magnetic viscosity

- $\eta$  = electrical conductivity  
 $c$  = velocity of light in vacuum  
 $\lambda, \mu$  = Lamé elastic constants  
 $\rho$  = density  
 $\alpha$  = coefficient of linear thermal expansion  
 $T_0$  = reference temperature of the body when unstressed  
 $C_v$  = specific heat at constant volume  
 $k_1 = \frac{k}{\rho C_v}$ , thermal diffusivity  
 $k$  = thermal conductivity  
 $Q$  = heat source  
 $F$  = body force component  
 $\delta(t)$  = Dirac delta function  
 $c_1^2 = \frac{\lambda + 2\mu}{\rho}$ ,  $b_1 = \frac{\beta}{\rho C_v}$ ,  $\beta = (3\lambda + 2)\alpha$   
 $d_1 = \frac{k_1}{T_0 \rho C_v c_1^2}$ ,  $a_1 = \frac{\beta T_0}{\lambda + 2\mu}$   
 $b = \frac{H_3^2}{\lambda + 2\mu}$ ,  $d = \frac{4\pi k_1}{c_1^2 \nu_H}$   
 $\epsilon = \frac{a_1 \beta}{\rho C_v}$

#### INTRODUCTION

In recent papers Bahar and Hetnarski<sup>1-3</sup>, have extended the state space approach developed by Bahar in boundary value problems<sup>4</sup> and heat conduction<sup>5</sup> to problems of coupled thermoelasticity. Das *et al.*<sup>6</sup> applied the state space approach to magneto-thermoelasticity as a further extension of Bahar and Hetnarski<sup>1</sup>. Das *et al.*<sup>7-10</sup>, also applied eigenvalue approach to the problems of thermoelasticity and magnetothermoelasticity. But in all these papers the authors have neglected the body forces in the equations of motion, the heat sources in the equation of heat conduction etc. As a result the vector-matrix differential equation involved becomes homogeneous. Recently Das *et al.*<sup>11</sup> have solved the problems of thermoelasticity and magneto-thermoelasticity by taking into account of the body forces and heat sources, and writing the equation in

an inhomogeneous vector-matrix differential equation. The solution has been made by state space approach.

In this paper we have attempted to solve the problem<sup>11</sup> by the eigenfunction expansion method. Theory for the solution of an inhomogeneous vector-matrix differential equation has been presented and for its applications we have chosen two particular problems of elasticity coupled with (i) a thermal field and (ii) a thermomagnetic field.

### THEORY

Consider the vector-matrix differential equation

$$\frac{d\mathbf{v}}{dx} = \mathbf{A} \mathbf{v} + \mathbf{f}(x) \quad \dots(1)$$

with the condition

$$\mathbf{v}(x_0) = \mathbf{c} \quad \dots(2)$$

where  $\mathbf{A}$  is an  $n \times n$  constant real matrix,  $\mathbf{c}$  is a given constant real  $n$ -vector and  $\mathbf{f}$  is a real  $n$ -vector function.

Let

$$\mathbf{v} = \mathbf{X} e^{\lambda x} \quad \dots(3a)$$

be a solution of the homogeneous equation

$$\frac{d\mathbf{v}}{dx} = \mathbf{A} \mathbf{v} \quad \dots(3b)$$

where  $\lambda$  is a scalar and  $\mathbf{X}$  is an  $n$ -vector independent of  $x$ . Substituting (3a) in (3b) we get

$$(\mathbf{A} \mathbf{X} - \lambda \mathbf{X}) e^{\lambda x} = 0 \Rightarrow \mathbf{A} \mathbf{X} - \lambda \mathbf{X} = 0 \Rightarrow \mathbf{A} \mathbf{X} = \lambda \mathbf{X}.$$

This may be interpreted that  $\lambda$  is an eigenvalue of the matrix  $\mathbf{A}$  and  $\mathbf{X}$  the corresponding right eigenvector.

Let  $\lambda_1, \lambda_2, \dots, \lambda_n$ , be  $n$  distinct eigenvalues of the matrix  $\mathbf{A}$  and  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$  be the corresponding right eigenvectors of the matrix  $\mathbf{A}$ . Then the vectors  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$  are linearly independent and so they form a basis of the space  $\Gamma^n$ , where  $\Gamma$  denotes the field of complex numbers. We can find scalars  $b_1, b_2, \dots, b_n$  such that

$$\mathbf{c} = b_1 \mathbf{X}_1 + b_2 \mathbf{X}_2 + \dots + b_n \mathbf{X}_n.$$

Choose

$$c_i = b_i e^{-\lambda_i x_0} \quad (i = 1, 2, \dots, n).$$

Let

$$\mathbf{u}(x) = \sum_{i=1}^n c_i \mathbf{X}_i e^{\lambda_i x}. \quad \dots(4a)$$

Thus  $u(x)$  is a solution of the differential equation (3b) and

$$u(x_0) = \sum_{i=1}^n c_i e^{\lambda_i x_0} \quad X_i = \sum_{i=1}^n b_i X_i = c.$$

Now, let

$$w(x) = \sum_{i=1}^n a_i(x) X_i e^{\lambda_i x} \quad \dots(4b)$$

be a solution of the differential equation (1), where  $a_1(x), a_2(x), \dots, a_n(x)$  are scalar functions of  $x$  such that  $a_i(x_0) = 0$ .

Differentiating (4b) with respect to  $x$ , we get

$$w'(x) = \sum_{i=1}^n a_i'(x) X_i e^{\lambda_i x} + \sum_{i=1}^n a_i(x) \lambda_i X_i e^{\lambda_i x}. \quad \dots(5)$$

Substituting (4b) and (5) in (1), we have

$$\sum_{i=1}^n a_i'(x) X_i e^{\lambda_i x} + \sum_{i=1}^n a_i(x) \lambda_i X_i e^{\lambda_i x} = \sum_{i=1}^n a_i(x) A X_i e^{\lambda_i x} + f(x)$$

or

$$\sum_{i=1}^n a_i'(x) X_i e^{\lambda_i x} = \sum_{i=1}^n a_i(x) [A X_i - \lambda_i X_i] e^{\lambda_i x} + f(x) = f(x). \quad \dots(6)$$

Multiplying (6) by  $Y_j e^{-\lambda_j x}$  [where  $Y_1, Y_2, \dots, Y_n$  are left eigenvectors of  $A$  corresponding to the eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ ] we get

$$\sum_{i=1}^n a_i'(x) Y_j X_i e^{(\lambda_i - \lambda_j)x} = Y_j f(x) e^{-\lambda_j x}$$

or

$$a_j'(x) Y_j X_j - Y_j f(x) e^{-\lambda_j x}, [\because Y_j X_i = 0 \text{ for } i \neq j]$$

or

$$a_j'(x) = \frac{1}{Y_j X_j} Y_j f(x) e^{-\lambda_j x}$$

or

$$a_j(x) = \int_{x_0}^x (Y_j X_j)^{-1} Y_j f(s) e^{-\lambda_j s} ds \quad \dots(7)$$

[∴  $a_j(x_0) = 0$  for  $j = 1, 2, \dots, n$ ].

Now take

$$v(x) = u(x) + w(x). \quad \dots(7a)$$

Differentiating we get

$$\begin{aligned} v'(x) &= u'(x) + w'(x) \\ &= A u(x) + A w(x) + f(x) \\ &= A [u(x) + w(x)] + f(x) \\ &= A v(x) + f(x) \end{aligned}$$

and

$$v(x_0) = u(x_0) + w(x_0) = c.$$

Hence  $v(x) = u(x) + w(x)$  is the unique solution of the differential equation (1) satisfying the condition (2).

### APPLICATIONS

We now proceed to apply the foregoing theory in the problems of thermoelastic and magnetothermoelastic interactions in an infinite medium.

#### I. *Thermoelastic interactions in an infinite solid with instantaneous heat source*

The coupled one dimensional equations of heat conduction and motion are given by

$$\left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) T = b_1 \frac{\partial^2 u}{\partial x \partial t} - d_1 Q(x, t) \quad \dots(8)$$

$$\left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) u = a_1 \frac{\partial T}{\partial x} - \frac{k_1}{c_1 (\lambda + 2\mu)} F(x, t). \quad \dots(9)$$

The equations are written in the non-dimensional form, where the position  $x$ , the displacement  $u$ , the normal stress  $\tau_{xx}$ , the time  $t$  and the temperature  $T$  are related to their dimensional counterparts  $x_1, u_1, (\tau_1)_{xx}, t_1$  and  $T_1$  by the relations  $x = c_1 x_1/k_1, u = c_1 u_1/k_1, \tau_{xx} = (\lambda + 2\mu)^{-1} (\tau_1)_{xx}, t = c_1^2 t_1/k_1$  and  $T = T_1/T_0$  respectively.

For the solution of eqns. (8) and (9), we shall use Laplace transforms defined by

$$\bar{u} = \int_0^\infty u \exp(-pt) dt, \quad \bar{T} = \int_0^\infty T \exp(-pt) dt.$$

Taking Laplace transforms of equations (8) and (9) and writing the resulting equations in the matrix form we obtain

$$\begin{aligned}
 \frac{d}{dx} \begin{bmatrix} \bar{T}(x, p) \\ \bar{u}(x, p) \\ \bar{T}'(x, p) \\ \bar{u}'(x, p) \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ p & 0 & 0 & b_1 p \\ 0 & p^2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} \bar{T}(x, p) \\ \bar{u}(x, p) \\ \bar{T}'(x, p) \\ \bar{u}'(x, p) \end{bmatrix} \\
 &+ \begin{bmatrix} 0 \\ 0 \\ -d_1 \bar{Q} \left( \frac{k_1}{c_1} x, \frac{k_1}{c_1^2} p \right) \\ -\frac{k_1}{c_1 (\lambda + 2\mu)} \bar{F} \left( \frac{k_1}{c_1} x, \frac{k_1}{c_1^2} p \right) \end{bmatrix} \dots(10)
 \end{aligned}$$

Where we have used the conditions that  $u, \frac{\partial u}{\partial t}$  and  $T$  are initially zero. The prime indicates differentiation with respect to  $x$ .

Equation (10) can be written in contracted form as

$$\frac{d\mathbf{X}}{dx} = \mathbf{A} \mathbf{v} + \mathbf{f}(x) \dots(11)$$

where  $\mathbf{v}(x, p)$  denotes the state vector in the transformed domain whose components consists of transformed temperature, displacement as well as their gradients.

We now consider an infinite medium which is unstressed and unstrained initially but has a uniform temperature distribution  $T_0$ . It is then subjected to heat sources distributed over a plane area. The problem is to determine the subsequent distribution of temperature, stress and deformations as well as the interaction between the temperature and deformation fields.

Let  $x_1 = \frac{k_1 x}{c_1} = 0$  represent the plane area over which the heat sources are situated and the solid medium occupies the whole space  $-a < x < +\infty$ . We assume that the heat source is instantaneous and acts on the plane  $r = 0$ . We may represent it as

$$Q \left( \frac{k_1}{c_1} x, \frac{k_1}{c_1^2} t \right) = q \delta \left( \frac{k_1 x}{c_1} \right) \delta \left( \frac{k_1 t}{c_1^2} \right)$$

So that

$$\bar{Q} \left( \frac{k_1}{c_1} x, \frac{k_1}{c_1^2} p \right) = q \frac{c_1^2}{k_1} \delta \left( \frac{k_1 x}{c_1} \right) \quad \dots(12)$$

where  $q$  is the constant strength of the source and  $\delta \left( \frac{k_1 x}{c_1} \right)$  is the Dirac delta function. The characteristic equation corresponding to the matrix  $A$  in equation (11) is given by

$$R^4 - p(p + 1 + \epsilon) R^2 + p^3 = 0 \quad \dots(13)$$

where  $\epsilon = a_1 b_1$  is the coupling parameter. Let the roots of the equation (13) be  $\pm R_1, \pm R_2$  which are also the eigenvalues of the matrix  $A$ . A set of right and left eigenvectors  $X_i, Y_i$  ( $i = 1, 2, 3, 4$ ) corresponding to the eigenvalues  $R_1, -R_1, R_2, -R_2$  of  $A$  can be calculated as

$$\begin{aligned} X_1 &= \begin{bmatrix} p^2 - R_1^2 \\ -a_1 R_1 \\ R_1 (p^2 - R_1^2) \\ -R_1^2 a_1 \end{bmatrix}, X_2 = \begin{bmatrix} p^2 - R_1^2 \\ a_1 R_1 \\ -R_1 (p^2 - R_1^2) \\ -R_1^2 a_1 \end{bmatrix} \\ X_3 &= \begin{bmatrix} p^2 - R_2^2 \\ -a_1 R_2 \\ R_2 (p^2 - R_2^2) \\ -R_2^2 a_1 \end{bmatrix}, X_4 = \begin{bmatrix} p^2 - R_2^2 \\ a_1 R_2 \\ -R_2 (p^2 - R_2^2) \\ -R_2^2 a_1 \end{bmatrix} \end{aligned} \quad \dots(14a)$$

$$\begin{aligned} Y_1 &= \left[ R_1^2 - p^2, p^2 b_1 R_1, \frac{R_1}{p} (R_1^2 - p^2), b_1 R_1^2 \right], \\ Y_2 &= \left[ R_1^2 - p^2, -p^2 b_1 R_1, -\frac{R_1}{p} (R_1^2 - p^2), b_1 R_2^2 \right], \\ Y_3 &= \left[ R_2^2 - p^2, p^2 b_1 R_2, \frac{R_2}{p} (R_2^2 - p^2), b_1 R_1^2 \right], \\ Y_4 &= \left[ R_2^2 - p^2, -p^2 b_1 R_2, -\frac{R_2}{p} (R_2^2 - p^2), b_1 R_2^2 \right] \end{aligned} \quad \dots(14b)$$

For simplicity we assume that the body force  $F(x, t) = 0$ , and assuming other physical conditions of the problem as in Das *et al.*<sup>11</sup> and Paria<sup>12</sup>, we write from (4a), (4b), (7), (7a), (12), (14a) and (14b)

$$v(x) = a_2(x) X_2 e^{-R_1 x} + a_4(x) X_4 e^{-R_2 x} \quad \dots(15)$$

where

$$a_2(x) = \frac{1}{Y_2 X_2} \int_{x_0}^x Y_2 f(a) e^{-R_1 s} ds,$$

$$= \frac{1}{Y_2 X_2} \frac{q c_1^3 d_1 R_1 (R_1^2 - p^2)}{p k_1^2}.$$

$$a_4(x) = \frac{1}{Y_4 X_4} \int_{x_0}^x Y_4 f(s) e^{-R_2 s} ds$$

$$= \frac{1}{Y_4 X_4} \frac{q c_1^3 d_1 R_2 (R_2^2 - p^2)}{p k_1^2}.$$

Thus the displacement and the temperature fields can be written from (15) as

$$\bar{u}(x, p) = \frac{q c_1^3 d_1 R_1^2 (R_1^2 - p^2) a_1}{p k_1^2 V_2} e^{-R_1 x} + \frac{q c_1^2 d_1 R_2^2 (R_2^2 - p^2) a_1}{p k_1^2 V_4}$$

$$\times e^{-R_2 x}, x > 0$$

$$\bar{T}(x, p) = - \frac{q c_1^2 d_1 R_1 (R_1^2 - p^2)^2}{p k_1^2 V_2} e^{-R_1 x} - \frac{q c_1^3 d_1 R_2 (R_2^2 - p^2)^2}{p k_1^2 V_4}$$

$$e^{-R_2 x}, x > 0$$

where

$$V_2 = - \left( R_1^2 - p^2 \right)^2 - a_1 R_1^2 b_1 \left( p^2 + R_1^2 \right) - \frac{R_1^2 (R_1^2 - p^2)^2}{p}$$

$$V_4 = - \left( R_2^2 - p^2 \right)^2 - a_1 R_2^2 b_2 \left( p^2 + R_2^2 \right) - \frac{R_2^2 (R_2^2 + p^2)^2}{d}.$$

Using the characteristic equation (13), one can write the expressions for  $\bar{u}(x, p)$  and  $\bar{T}(x, p)$  in a simplified form

$$\bar{u}(x, p) = \frac{q a_1 c_1^3 d_1}{2 k_1^2 (R_1^2 - R_2^2)} [e^{-R_2 x} - e^{-R_1 x}], x > 0. \quad \dots(16)$$

$$\bar{T}(x, p) = \frac{q c_1^3 d_1}{2 k_1^2 (R_1^2 - R_2^2) R_1 R_2} \left[ R_2 \left( R_1^2 - p^2 \right) e^{-R_1 x} - R_1 \left( R_2^2 - p^2 \right) \right.$$

$$\left. \times e^{-R_2 x} \right], x > 0. \quad \dots(17)$$

Equations (16) and (17) are in complete agreement with Das *et al.*<sup>11</sup> and these equations, when written in the dimensional form, are also in complete agreement with Paria<sup>12</sup> in different notations.

*II Magnetothermoelastic interactions in an infinite solid with instantaneous heat source*

In the absence of displacement current and charge density, Maxwell's equations reduce to [Vide Paria<sup>13</sup>]

$$\text{rot } \mathbf{H} = \frac{4\pi}{c} \mathbf{j}, \text{ rot } \mathbf{E} = - \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t_1} \quad \dots(18)$$

$$\text{div } \mathbf{B} = 0, \mathbf{B} = \mu_e \mathbf{H},$$

while Ohm's law is given by

$$\mathbf{j} = \eta \left[ \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{u}_1}{\partial t_1} \times \mathbf{B} \right], \quad \dots(19)$$

where we have neglected the effect of temperature on current.

The equation of heat conduction, neglecting the effect of current on temperature, is given by

$$k \nabla^2 T_1 + Q = \rho C_v \frac{\partial T_1}{\partial t_1} + T_0 \beta \frac{\partial^2 u_1}{\partial x_1 \partial t_1}. \quad \dots(20)$$

Taking into account the Lorentz body force, the equation of motion becomes,

$$\rho \frac{\partial^2 (u_1)_i}{\partial t_1^2} = \frac{\partial (\tau_1)_{ik}}{\partial (x_1)_k} + \frac{1}{c} (\mathbf{j} \times \mathbf{B})_i + F. \quad \dots(21)$$

The stress-strain relations are given by

$$(\tau_1)_{ij} = 2\mu (e_1)_{ij} - (\lambda e_1 - \beta T_1) \delta_{ij}$$

where

$$(e_1)_{ij} = \frac{1}{2} \{ (u_1)_{i,j} + (u_1)_{j,i} \}$$

$$e_1 = \text{div } (\mathbf{u}_1).$$

[subscript "1" indicates the quantities involved are dimensional].

Assuming for simplicity  $\mu_e \approx 1$  (as is practically justified) the coupled one dimensional equations of magnetothermoelasticity in the non-dimensional form were given by Das and Bhattacharya<sup>6</sup>, Paria<sup>13</sup> as

$$\left( \frac{\partial^2}{\partial x^2} - d \frac{\partial}{\partial t} \right) h_y = 0 \quad \dots(22)$$

$$\left( \frac{\partial^2}{\partial x^2} - d \frac{\partial}{\partial t} \right) \left( \frac{h_z}{H_0} \right) = Q \frac{\partial^2 u}{\partial x \partial t} \quad \dots(22)$$

$$\left( \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t} \right) T = b_1 \frac{\partial^2 u}{\partial x \partial t} - d_1 q \left( \frac{k_1}{c_1} x, \frac{k_1}{c_1^2} t \right) \quad \dots(24)$$

$$\left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) u = a_1 \frac{\partial T}{\partial x} + b \frac{\partial}{\partial x} \left( \frac{h_x}{H_3} \right) - \frac{k_1}{c_1 (\lambda + 2\mu)}$$

$$\times F \left( \frac{k_1}{c_1} x, \frac{k_1}{c_1^2} t \right). \quad \dots(25)$$

Equation (22) gives  $h_y = 0$  since  $h_y$  is zero initially. Now for the solutions of eqns. (23)-(25) for the variables  $u$ ,  $T$  and  $h_x$  we proceed as follows.

We consider an infinite elastic solid without any initial stress or strain but having an uniform temperature distribution  $T_0$  throughout. The body is embedded in a uniform magnetic field of intensity  $H_3$ . The problem is to determine the distribution of stress, strain, temperature electric and magnetic field in the body, if it is subjected to instantaneous heat sources distributed on a plane extending in all directions. Without any loss of generality, the plane containing the heat sources can be taken parallel to the initial magnetic field of intensity  $H_3$ , and  $y_1$ -axis in the plane of the heat sources. The axis of  $x_1$  is then perpendicular to this plane such that  $x_1, y_1, z_1$  axis form a right handed system of cartesian co-ordinates. We take the heat source in the same form as in (12) and neglect the body forces. For simplicity, we assume that the medium is of infinite conductivity.

Letting  $\eta \rightarrow \infty$  i. e.  $v_H \rightarrow 0$  or  $d \rightarrow \infty$ , appropriate equations from (23)-(25) reduce, in Laplace transform domain, to

$$\bar{h}_x = -H_3 \frac{d\bar{u}}{dx} \quad \dots(26)$$

$$\left( \frac{d^2}{dx^2} - p \right) \bar{T} = b_1 p \frac{d\bar{u}}{dx} - d_1 \bar{Q} \left( \frac{k_1}{c_1} x, \frac{k_1}{c_1^2} p \right) \quad \dots(27)$$

$$\left( \frac{d^2}{dx^2} - p^2 \right) \bar{u} = a_1 \frac{d\bar{T}}{dx} + \frac{b}{H_3} \frac{d\bar{h}_x}{dx} \quad \dots(28)$$

Using  $\bar{h}_x$  from (26) in (28), the resulting eqn. and the equation (27) can be written in the matrix form as

$$\frac{d}{dx} \begin{bmatrix} \bar{T}(x, p) \\ \bar{u}(x, p) \\ \bar{T}'(x, p) \\ \bar{u}'(x, p) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ p & 0 & 0 & b_1 p \\ 0 & \frac{p^2 a_1}{1+b} & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{T}(x, p) \\ \bar{u}(x, p) \\ \bar{T}'(x, p) \\ \bar{u}'(x, p) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ -d_1 \bar{Q} \left( \frac{k_1}{c_1} x, \frac{k_1}{c_1^2} p \right) \\ 0 \end{bmatrix} \quad \dots(29)$$

which can be written in the contracted form as

$$\frac{d\mathbf{v}}{dx} = \mathbf{A} \mathbf{v} + \mathbf{f}(x). \quad \dots(30)$$

The characteristic equation corresponding to the matrix  $\mathbf{A}$  in (30) is given by

$$R^4 - \frac{p(p+1+\epsilon+b)}{1+b} R^2 + \frac{p^3}{1+b} = 0. \quad \dots(31)$$

Let  $\pm R_1, \pm R_2$  be roots of the equation (31), then  $R_1, -R_1, R_2$  and  $-R_2$  are the eigenvalues of  $\mathbf{A}$ .

A set of right and left eigenvectors  $X_i, Y_i$  ( $i = 1, 2, 3, 4$ ) corresponding to the eigenvalues  $R_1, -R_1, R_2, -R_2$  of  $\mathbf{A}$  can be calculated as

$$\begin{aligned} X_1 &= \left[ \begin{array}{c} -R_1^2 + \frac{p^2}{1+b} \\ \frac{-a_1 R_1}{1+b} \\ R_1 \left( -R_1^2 + \frac{p^2}{1+b} \right) \\ \frac{-a_1 R_1^2}{1+b} \end{array} \right] & X_2 &= \left[ \begin{array}{c} -R_1^2 + \frac{p^2}{1+b} \\ \frac{a_1 R_1}{1+b} \\ -R_1 \left( -R_1^2 + \frac{p^2}{1+b} \right) \\ \frac{-a_1 R_1^2}{1+b} \end{array} \right] \\ X_3 &= \left[ \begin{array}{c} -R_2^2 + \frac{p^2}{1+b} \\ \frac{-a_1 R_2}{1+b} \\ R_2 \left( -R_2^2 + \frac{p^2}{1+b} \right) \\ \frac{-a_1 R_2^2}{1+b} \end{array} \right] & X_4 &= \left[ \begin{array}{c} -R_2^2 + \frac{p^2}{1+b} \\ \frac{a_1 R_2}{1+b} \\ -R_2 \left( -R_2^2 + \frac{p^2}{1+b} \right) \\ \frac{-a_1 R_2^2}{1+b} \end{array} \right] \end{aligned} \quad \dots(32a)$$

$$\begin{aligned} Y_1 &= \left[ R_1^2 - \frac{p^2}{1+b}, \frac{b_1 p^2 R_1}{1+b}, \frac{R_1}{p} \left( R_1^2 - \frac{p^2}{1+b} \right), b_1 R_1^2 \right], \\ Y_2 &= \left[ R_1^2 - \frac{p^2}{1+b}, \frac{-b_1 p^2 R_1}{1+b}, -\frac{R_1}{p} \left( R_1^2 - \frac{p^2}{1+b} \right), b_1 R_1^2 \right] \\ Y_3 &= \left[ R_2^2 - \frac{p^2}{1+b}, \frac{b_1 p^2 R_2}{1+b}, \frac{R_2}{p} \left( R_2^2 - \frac{p^2}{1+b} \right), b_1 R_2^2 \right] \\ Y_4 &= \left[ R_2^2 - \frac{p^2}{1+b}, \frac{-b_1 p^2 R_2}{1+b}, -\frac{R_2}{p} \left( R_2^2 - \frac{p^2}{1+b} \right), b_1 R_2^2 \right] \end{aligned} \quad \dots(32b)$$

Assuming the physical conditions to be the same as in Paria<sup>13</sup> and Das *et al.*<sup>11</sup> of the problem, we write from (4a), (4b), (7), (7a), (12), (32a) and (32b);

$$v(x) = a_2(x) X_2 e^{-R_1 x} + a_4(x) X_4 e^{-R_2 x} \quad \dots(33)$$

where

$$\begin{aligned} a_2(x) &= \frac{1}{Y_2 X_2} \int_{x_0}^x Y_2 f(s) e^{-R_1 s} ds, \quad x_0 > 0 \\ &= \frac{1}{Y_2 X_2} \frac{q c_1^3 d_1 R_1 \left( R_1^2 - \frac{p^2}{1+b} \right)}{p k_1^2} \end{aligned}$$

$$\begin{aligned} a_4(x) &= \frac{1}{Y_4 X_4} \int_{-\infty}^x Y_4 f(s) e^{-R_2 s} ds \\ &= \frac{1}{Y_4 X_4} \frac{q c_1^2 d_1 R_2 \left( R_2^2 - \frac{p^2}{1+b} \right)}{p k_1^2}, \quad x > 0 \end{aligned}$$

and

$$\begin{aligned} Y_2 X_2 &= - \left( R_1^2 - \frac{p^2}{1+b} \right)^2 \left( 1 + \frac{R_1^2}{p} \right) - \frac{a_1 b_1 R_1^2}{1+b} \left( \frac{p^2}{1+b} + R_1^2 \right), \\ Y_4 X_4 &= - \left( R_2^2 - \frac{p^2}{1+b} \right)^2 \left( 1 + \frac{R_2^2}{p} \right) - \frac{a_1 b_1 R_2^2}{1+b} \left( \frac{p^2}{1+b} + R_2^2 \right). \end{aligned}$$

As in the previous application we now use the characteristic equation (31) and write the displacement and temperature fields from (33) in a simplified form as follows :

$$\bar{u}(x, p) = \frac{q c_1^3 d_1 a_1}{2 k_1^2 (R_1^2 - R_2^2) (1+b)} (e^{-R_2 x} - e^{-R_1 x}), \quad x > 0 \quad \dots(34)$$

$$\begin{aligned} \bar{T}(x, p) &= \frac{q c_1^3 d_1}{2 k_1^2 R_1 R_2 (R_1^2 - R_2^2)} \left[ R_2 \left( R_1^2 - \frac{p^2}{1+b} \right) e^{-R_1 x} \right. \\ &\quad \left. - R_1 \left( R_2^2 - \frac{p^2}{1+b} \right) e^{-R_2 x} \right], \quad x > 0. \quad \dots(35) \end{aligned}$$

The perturbation of the magnetic field can now be determined from (26) as

$$\bar{h}_x(x, p) = - \frac{H_3 q c_1^3 d_1 a_1}{2 (1+b) k_1^2 (R_1^2 - R_2^2)} [R_1 e^{-R_1 x} - R_2 e^{-R_2 x}], \quad x > 0 \quad \dots(36)$$

It may be noticed that in the absence of the magnetic field i.e. when  $b = 0$ , eqns. (34) and (35) are identical with (16) and (17).

Paria<sup>12</sup> solved the same problem by the method of Laplace-Fourier transforms and found out the expression for  $\bar{h}_z(x, p)$  in eqn. (10.36a) in the Laplace transform domain, but this expression is not in agreement with our expression (36) due to a mistake in his calculation in eqn. (10.15d).

#### 4. NUMERICAL INVERSION AND ANALYSIS OF THE RESULTS

The numerical inversion in the space time domain of the displacement and temperature fields of the problem in application I i. e. the inversions of eqns. (16) and (17) have been made by Paria<sup>12</sup> for small time. Paria<sup>13</sup> and Das *et al.*<sup>11</sup>. presented the distribution of stress and induced magnetic field for small time of the problem in application II.

We now propose to invert numerically the temperature fields for both the problems in applications I and II by Bellman method for any time interval. Although we have made a particular choice of the time interval and compared the magnetic effect upon the temperature distribution in the two problems, the comparison can be made for any time interval by the shifting theorem of Laplace transform, vide Bellman<sup>14</sup>.

In order to compare the interactions of the magnetic field with the thermoelastic field is the deformation and temperature; we now propose the medium to be copper, and thus  $\epsilon = 0.003722$  in MKS units, and take  $x = 1$ , and invert eqns. (16), (17), (34) and (35) for a specific time range and for two different values of the intensity of the magnetic field as shown in the table. Finally, the inversion for the perturbation of the magnetic field,  $h_z$  in eqn. (36), has been made for the same two values of the intensities for comparison. The computations have been made in Burroughs 6700 of the Regional Computer Centre, Jadavpur University, Calcutta.

It is clear from Table I that the displacement, temperature and the perturbation of the original magnetic field have the small oscillatory character about the time axis and they all decrease with the intensity of the magnetic field, as expected. It is also clear that the effect of the magnetic field is not much pronounced on the temperature field in comparison to the displacement field and the perturbation of the original magnetic field.

#### CONCLUDING REMARKS

It is clear from foregoing analysis that the present approach can be very easily applied to a broad class of boundary value problems in mechanics and other fields where solution of simultaneous, coupled, differential equations is required. The state space methodology of Bahar and Hetnarski<sup>1</sup> for homogeneous equations and of Das *et al.*<sup>11</sup> for non-homogeneous equations converts the boundary value problem to an initial value problem before its solution. Thus the final solution requires very extensive use of algebra for the determination of these initial values through boundary condition by using the matrix equations of the influence functions. In the present approach, the original structure of the problem is retained<sup>6</sup>.

TABLE I

In the table the numerical values, as obtained from the computer, are rounded off to six significant figures

Eqn. no.	$t$	0.0257750	0.138382	0.352509	0.693147	1.21376	2.04612	5.67119
16	$2k_1^2 u(1,t)$	-0.182549E+05	0.256421E+05	-0.293849E+05	0.319915E+05	-0.354164E+05	0.423166E+05	-0.626672E+5
	$qa_1^3 c_1^3 d_1$							
34	$b = 0$							
	$2k_1^2 u(1,t)$	-0.174558E+05	0.245012E+05	-0.280516E+05	0.305143E+05	-0.337593E+05	0.403193E+05	-0.596955E+05
	$qc_1^3 d_1 a_1$							
34	$b = 0.01$							
	$2k_1^2 u(1,t)$	-0.119751E+05	0.166968E+05	-0.189622E+05	0.204774E+05	-0.225301E+05	0.268085E+05	-0.396133E+05
	$qc_1^3 d_1 a_1$							
17	$b = 0.1$							
	$2k_1^2 T(1,t)$	-0.423791E+03	0.4117128+03	-0.348171E+03	0.315162E+03	-0.315661E+03	0.355571E+03	-0.513241E+03
	$qc_1^3 d_1$							
35	$b = 0$							
	$2k_1^2 T(1,t)$	-0.420903E+03	0.407584E+03	-0.343340F+03	0.309802E+03	-0.309644E+03	0.348315E+03	-0.502442E+03
	$qc_1^3 d_1$							
35	$b = .01$							
	$2k_1^2 T(1,t)$	-0.400634E+03	0.378679E+03	-0.309623E+03	0.272523E+03	-0.267902E+03	0.298063E+03	-0.427729E+03
	$qc_1^3 d_1$							
	$b = 0.1$							

(Table continued on p. 711)

Eqn. no.	$t$	0.0257750	0.138382	0.352509	0.693147	1.21376	2.04612	3.67119
36	$\frac{2k_1^2 h_2(1,t)}{H_3 q c_1^2 d_1 a_1}$	0.160832E+06	-0.229348E+06	0.2676 E+06	-0.296158E+06	0.331901E+06	0.399815E+06	0.594666E+06
	$b = 0.01$							
36	$\frac{2k_1^2 h_2(1,t)}{H_3 q c_1^2 d_1 a_1}$	0.106986E+06	-0.151768E+06	0.179908E+06	-0.193395E+06	0.215633E+06	0.258852E+06	0.384274E+06
	$b = 0.1$							

We hope to communicate, in future, the theory and application for the case when the eigenvalues of the matrix  $A$  in (1) are repeated.

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