MAGNETO-ELASTIC TRANSVERSE SURFACE WAVES IN SELF-REINFORCED ELASTIC SOLIDS

P. D. S. Verma, O. H. Rana and Meenu Verma

Department of Mathematics, Regional Engineering College, Kurukshetra 1132119

(Received 20 May 1987; after revision 7 December 1987)

When the direction of a uniform magnetic field is different from that of wave propagation, the surface wave of the SH type can also be propagated in a self reinforced, anisotropic elastic half space without dispersion, as in isotropic case. Reinforcement happens only to increase the decay rate of the waves.

1. INTRODUCTION

Idea, here, is to examine whether it is possible to propagate the uncoupled surface waves of SH type without showing dispersion\(^1\) in a perfectly conducting anisotropic elastic half space, endowed with self reinforcement\(^2\) in the presence of a uniform magnetic field or not. We find, as in elastic case\(^3\), that when the plane of the magnetic field and the direction of wave propagation is normal to free surface, it is possible provided the electromagnetic radiation into the adjacent free space is not neglected. The role of reinforcement is merely to increase the decay rate of the waves.

2. BASIC EQUATIONS

(i) Stress-strain law governing self reinforced elastic medium\(^2\) whose preferred direction is that of a unit vector \(\mathbf{a}\) is given by

\[
t_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} + \alpha (a_k \varepsilon_{am} \varepsilon_{km} \delta_{ij} + \varepsilon_{kk} a_i a_j) \\
+ 2 (\mu_L - \mu_T) (a_i \varepsilon_{kij} + a_j \varepsilon_{akj}) + \beta a_k \varepsilon_{am} \varepsilon_{km} a_i a_j.
\]

\(2.1\)

In (2.1), \(\lambda, \mu_T, \mu_L, \beta\) and \(\alpha\) are all elastic constants with the dimensions of stress. \(t_{ij}\) is stress tensor, \(\varepsilon_{ij}\) is the strain tensor. \(a_i\) are the components of \(\mathbf{a}\), the unit vector denoting the preferred direction. All suffixes take values from 1 to 3 and repeated suffix means summation. We are taking the physical components of the respective tensors. \(\delta_{ij}\) is the kronecker delta.

(ii) The electromagnetic field equations for free space are

\[
\nabla^2 \mathbf{h} - \frac{1}{C^2} \frac{\partial^2 \mathbf{h}}{\partial t^2} = 0, \quad \text{Curl} \mathbf{h} = \varepsilon_0 \frac{\partial \mathbf{e}}{\partial t} \\
\nabla^2 \mathbf{e} - \frac{1}{C^2} \frac{\partial^2 \mathbf{e}}{\partial t^2} = 0, \quad \text{Curl} \mathbf{e} = -\mu_0 \frac{\partial \mathbf{h}}{\partial t}.
\]

\(2.2\)

\(2.3\)
In (2.2) to (2.3) $e$ and $h$ are perturbations in the electric and magnetic field vectors within the medium, given by

$$ e = - \mu_e \left( \frac{\partial u}{\partial t} \times H \right) \quad h = \text{Curl} \left( u \times H \right) $$

...(2.4)

where $u$ is the displacement vector, $H$ is the primary magnetic field and $\mu_e$ is the magnetic permeability. In (2.2) and (2.3) $C = (\varepsilon_0 \mu_0)^{-1/2}$ denotes the velocity of light, $\varepsilon_0$ and $\mu_0$ being the electric and magnetic permeabilities of the free space.

(iii) The equation of motion is

$$ t_{ij,j} + (J \times B)_i = \rho \frac{\partial^2 u_i}{\partial t^2}. $$

...(2.5)

In (2.5), $u_i$ are the components of displacement $u$, $J$ and $B$ are current density and magnetic induction vectors respectively. We have

$$ B = \mu_e H, \quad J = \sigma \left[ E + \frac{\partial u}{\partial t} \times B \right]. $$

...(2.6)

(iv) Boundary conditions to be satisfied at the free boundaries of separation are

$$ \left[ e + \frac{\partial u}{\partial t} \times B \right]_t = 0, \quad \left[ B \right]_n = 0 $$

...(2.7)

$$ (t_{ij} + M_{ij}) v_j = \overline{M}_{ij} v_j, $$

...(2.8)

where $M_{ij}$ and $\overline{M}_{ij}$ are the Maxwell stress tensors for the material and for the free space respectively, $v_j$ is the unit normal vector at a point of the boundary. A double bracket with a suffix $t$ and $n$ in (2.7) means difference between the tangential and the normal components.

3. Surface Waves of SH Type

We consider the propagation of harmonic waves in a self-reinforced anisotropic elastic medium occupying the semi-infinite space $Z \geq 0$ (the plane $Z = 0$ which separates the medium from the free space, is free of mechanical tractions) in the presence of a uniform magnetic field. We take the displacement and magnetic field components in the form

$$ u = 0, \quad v = A \exp \left( pz + ik (x - Vt) \right), \quad \omega = 0 $$

...(3.1)

$$ H_x = H \cos \phi, \quad H_y = 0, \quad H_z = H \sin \phi, $$

...(3.2)

$\phi$ being the angle at which the wave crosses the magnetic field. The components of reinforcement director are

$$ a = (a_1, 0, a_3). $$

...(3.3)

For free space, the elastic and magnetic field perturbations are

$$ e = (e_1, 0, e_3) e^{2\pi ik(e-Vt)} $$

...(3.4)
\[ h = (0, \bar{h}_2, 0) e^{az} + ik (x - Vt) \] \hspace{2cm} \ldots (3.5)

the constants \( \bar{a} \) and \( \bar{h} \) being connected by

\[ \bar{a}_1 = -\frac{iq}{\varepsilon_0 k V} \bar{h}_2, \quad \bar{a}_3 = -\frac{1}{\varepsilon_0 V} \bar{h}_2. \hspace{2cm} \ldots (3.6) \]

In (3.4) to (3.6), \( q \) being given by

\[ q^2 = k^2 \left( 1 - \frac{V^2}{C^2} \right) \hspace{2cm} \ldots (3.7) \]

is to be taken with proper sign so that electromagnetic radiation into free space does not vanish for \( Z \to -\infty \).

4. Solution

The equations determining \( v \) and \( h_2 \) are

\[ \begin{bmatrix} p H \sin \phi + i k H \cos \phi & -1 \\ p^2 \beta^2 + k^2 (V^2 - \beta^2) k/\rho (p H \sin \phi + i k H \cos \phi) \end{bmatrix} \begin{bmatrix} v \\ h_2 \end{bmatrix} = 0. \hspace{2cm} \ldots (4.1) \]

Equation in \( p \) becomes

\[ \begin{bmatrix} \beta^2 + V_A^2 \sin^2 \phi + (\mu_L - \mu_T) \frac{1}{\rho} a_3^2 \end{bmatrix} p^2 + i k p \left[ V_A^2 \sin 2\phi \right. \\
+ 2a_1 a_3 \frac{\mu_T - \mu_T}{\rho} + k^2 [V^2 - \beta^2 - V_A^2 \cos^2 \phi] - a_1^2 \left( \mu_L - \mu_T \right) = 0 \hspace{2cm} \ldots (4.2) \]

where \( \beta = \sqrt{\mu_T/\rho} \) is the velocity of plane shear waves and \( V_A = \sqrt{\mu_T/\rho} \) \( H \) is the Alfvén velocity. Out of the two roots of (4.2), we choose one with a negative real part as for surface waves, displacement decays with increasing depth.

Equation (2.7) gives

\[ \bar{a}_1 = ik V \mu_e H \sin \phi v, \hspace{0.5cm} Z = 0. \hspace{2cm} \ldots (4.3) \]

The condition that the surface \( Z = 0 \) is free from mechanical tractions reduces to

\[ p \left[ \beta^2 + V_A^2 \sin^2 \phi + \frac{\mu_L - \mu_T}{\rho} a_3^2 + ik \left( \frac{1}{2} V_A^2 \sin 2\phi + \frac{\mu_L - \mu_T}{\rho} a_1 a_3 \right) \right. \\
+ V_A^2 \frac{V^2 \sin^2 \phi}{q c^2} = 0, \hspace{0.5cm} V_A^2 = \frac{\mu_e H^2}{\mu_0 \rho} \hspace{2cm} \ldots (4.4) \]

Elimination of \( p \) between (4.2) and (4.4) gives us

\[ V_A^4 \left[ 1 - \frac{V_A^4 \sin^4 \phi}{C^2 I} \right] - V^2 \left[ V^2 + \frac{O}{I} \right] + C^2 \frac{O}{I} = 0 \hspace{2cm} \ldots (4.5) \]
where
\[
Q = \beta^2 \left( \beta^2 + V_A^2 \right) + \{(\mu_L - \mu_T)/\rho \} \left\{ \beta^2 + V_A^2 (a_1 \sin \phi - a_2 \cos \phi)^2 \right\}
\]
\[\ldots (4.6)\]
\[
I = V_A^2 \sin^2 \phi + \beta^2 + \{(\mu_L - \mu_T)/\rho \} a_3^2 .
\]
\[\ldots (4.7)\]

To first approximation in \(1/C\), the roots of (4.5) are
\[
V_1^2 = C^2 \left( 1 - \frac{V_A^4 \sin^4 \phi}{C^2 I} \right), \quad V_2^2 = \frac{Q}{I} \left( 1 - \frac{V_A^4 \sin^4 \phi}{C^2 I^2} \right).
\]
\[\ldots (4.8)\]

5. DISCUSSION OF THE RESULTS

(i) For \(V_1, q\) becomes imaginary and \(p\) has zero real part as is clear from equation (4.4). Hence this value is not suitable. The second value \(V_2\) satisfies all the conditions. Thus purely transverse surfaces waves can be propagated, without any dispersion, in a conducting elastic but anisotropic half space, provided there exists an external magnetic field non aligned to the direction of wave propagation. Waves propagate with velocity \(V_2\) given by (4.8). Penetration depth is given by
\[
\left\{ d = \left\{ qC^2 V_A^2 \sin^2 \phi + \beta^2 + (\mu_L - \mu_T)/\rho a_3^2 \right\} \right\} \frac{V_2^2}{V^2} k \sin^2 \phi.
\]

(ii) The reinforcement increases the strength of the transverse surface waves which decay out more quickly. Energy of the wave gets dissipated within a shorter longitudinal span.

For infinite thickness and conductivity of the plate, the velocity of the wave obtained in Verma becomes the same as the one derived from the present solution on the assumption of infinite conductivity.

REFERENCES