ON MINIMAL PAIRWISE HAUSDORFF BITOPOLOGICAL SPACES

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The purpose of this paper is to investigate minimal pairwise Hausdorff bitopological spaces. The pairwise semiregularization of a bitopological space is introduced and a brief outline of its properties is given. A new concept of pairwise almost compactness is defined and filter characterizations of (pairwise Hausdorff) pairwise almost compact spaces are obtained. The above concepts are used to prove bitopological analogues of two well-known Theorems given by M. Katetov and B. Banaschewski for minimal Hausdorff topological spaces.

INTRODUCTION

The concept of minimal pairwise Hausdorff bitopological spaces was initiated by Raghavan and Reilly. The purpose of this paper is to continue the study of the above concept.

In section 1 we introduce the pairwise semiregularization of a bitopological space and examine its properties. We investigate also a certain type of bitopological adherence and convergence of filters. In section 2 we introduce a new concept of pairwise almost compactness and discuss its relationship to other well-known forms of bitopological compactness. In section 3 we give filter characterizations of (pairwise Hausdorff) pairwise almost compact spaces. In section 4 we improve a characterization of minimal pairwise Hausdorff spaces given by Raghavan and Reilly, and generalize two well-known Theorems of Katetov and Banaschewski. Finally, we prove that the concept of minimal pairwise Hausdorff spaces is not product invariant.

Throughout this paper if \((X, \tau_i, \tau_j)\) is a bitopological space the \(\tau_i\)-interior of a set \(A \subseteq X\) is denoted by \(<A>_{\tau_i}\) or \(\langle A\rangle_{\tau_i}\) and the \(\tau_i\)-closure of \(A\) by \([A]_{\tau_i}\), where \(i = 1, 2\). The set of all \(\tau_i\)-neighbourhoods of a point \(x \in X\) is denoted by \(\mathcal{H}_i(x)\) and the set of all \(\tau_i\)-open neighbourhoods of \(x\) by \(\tau_i(x), i = 1, 2\). Whenever we deal with a statement involving the topologies \(\tau_i\) and \(\tau_j\) it will be understood that \(i, j = 1, 2\) and \(i \neq j\). Generally terms and notations not explained in this paper are those of Kelly, Cooke and Reilly and Raghavan and Reilly.

1. PRELIMINARY DEFINITIONS AND THEOREMS

A. Pairwise Semiregularization

Let \((X, \tau_i, \tau_j)\) be a bitopological space. According to Singal and Singal a subset \(A\) of \(X\) is said to be \((i, j)\)-regularly open if \(A = <[A]_{\tau_i}, >_{\tau_j}\).
Definition 1.1—\((X, \tau_1, \tau_2)\) is said to be \((i, j)\)-semiregular if for each \(x \in X\) the collection of all \((i, j)\)-regularly open neighbourhoods of \(x\) is a \(\tau_i\)-open neighbourhood base at \(x\). If \((X, \tau_1, \tau_2)\) is \((1, 2)\)-and \((2, 1)\)-semiregular it is called pairwise semiregular (henceforth abbreviated as \(p\)-semiregular).

Since the intersection of two \((i, j)\)-regularly open sets is an \((i, j)\)-regularly open set, we obtain the following result.

Proposition 1.2—The collection of all \((i, j)\)-regularly open sets forms a base for a topology \(\tau_i^s\) on \(X\), coarser than \(\tau_i\).

Definition 1.3—The bitopological space \((X, \tau_1^s, \tau_2^s)\) is called the pairwise semi-regularization (henceforth abbreviated as \(p\)-semi-regularization) of \((X, \tau_1, \tau_2)\).

Theorem 1.4—A bitopological space \((X, \tau_1, \tau_2)\) is \((i, j)\)-semiregular (resp. \(p\)-semi-regular) iff \(\tau_i^s = \tau_i\) (resp. \(\tau_i^s = \tau_1\) and \(\tau_2^s = \tau_2\)).

Proof: Follows from Definition 1.1 and Proposition 1.2.

Using the fact that the set \([A]_j\) is \((i, j)\)-regularly open for each \(A \subseteq X\), we can easily prove the following lemma.

Lemma 1.5—If \((X, \tau_1^s, \tau_2^s)\) is the \(p\)-semi-regularization of \((X, \tau_1, \tau_2)\), then \([V]_j = [V]_{\tau_j^s}\) and \([A]_j = [A]_{\tau_j^s}\) for each \(V \in \tau_i\).

Proposition 1.6—A subset \(A\) of \(X\) is \((i, j)\)-regularly open in \((X, \tau_1, \tau_2)\) iff it is \((i, j)\)-regularly open in \((X, \tau_1^s, \tau_2^s)\).

Proof: Follows from Lemma 1.5.

Theorem 1.7—The \(p\)-regularization \((X, \tau_1^s, \tau_2^s)\) of \((X, \tau_1, \tau_2)\) is a \(p\)-semi-regular bitopological space.

Proof: Follows from Propositions 1.2, 1.6 and Theorem 1.4.

Our last result is easily proved.

Theorem 1.8—\((X, \tau_1, \tau_2)\) is \(p\)-Hausdorff iff \((X, \tau_1^s, \tau_2^s)\) is \(p\)-Hausdorff.

B. \(\theta_{ij}\)-adherence and convergence of filters

Let \((X, \tau_1, \tau_2)\) be a bitopological space.

Definition 1.9—A point \(x \in X\) is said to be \(\theta_{ij}\)-adherent point of \(A \subseteq X\) if \([V]_j \cap A \neq \emptyset\) for each \(V \in \mathcal{A}_i(x)\). The set of all \(\theta_{ij}\)-adherent points of \(A\) is denoted by \([A]_j\).
Definition 1.10—A point \( x \in X \) is said to be \( \theta_{ij} \)-limit point (resp. \( \theta_{ij} \)-adherent point) of a filter \( \mathcal{F} \) on \( X \) if \( \mathcal{F} \supset \{[V]_{ij} : V \in \mathcal{F}_i(x)\} \) (resp. if \( x \in \cap \{[F]_{ij} : F \in \mathcal{F}\} \)). If \( \mathcal{B} \) is a filterbase on \( X \), then a point \( x \in X \) is called \( \theta_{ij} \)-limit point (resp. \( \theta_{ij} \)-adherent point) of \( \mathcal{B} \) if \( x \) is \( \theta_{ij} \)-limit point (resp. \( \theta_{ij} \)-adherent point) of the filter \( \mathcal{F} \) generated by \( \mathcal{B} \). The set of all \( \theta_{ij} \)-limit points of a filter \( \mathcal{F} \) (resp. filterbase \( \mathcal{B} \)) is denoted by \( \theta_{ij} \)-lim \( \mathcal{F} \) (resp. \( \theta_{ij} \)-lim \( \mathcal{B} \)) and the set of all \( \theta_{ij} \)-adherent points of a filter \( \mathcal{F} \) (resp. filterbase \( \mathcal{B} \)) by \( \theta_{ij} \)-adh \( \mathcal{F} \) (resp. \( \theta_{ij} \)-adh \( \mathcal{B} \)).

It is obvious that a point \( x \in X \) is \( \theta_{ij} \)-limit point (resp. \( \theta_{ij} \)-adherent point) of a filter base \( \mathcal{B} \) on \( X \) iff for each \( V \in \mathcal{F}_i(x) \) there exists a \( B \in \mathcal{B} \) such that \( B \subset [V]_{ij} \) (resp. iff \( x \in \cap \{[B]_{ij} : B \in \mathcal{B}\} \)).

We recall that a filter \( \mathcal{F} \) on \( X \) is called \( \tau \)-open if it has a base \( \mathcal{B} \) consisting exclusively by \( \tau \)-open sets.

The proofs of the following results are straightforward and therefore they are omitted.

Proposition 1.11—(a) For each \( A \subset X \), \([A]_i \subset [A]_{ij} \).

(b) If \( A \subset B \subset X \), then \([A]_{ij} \subset [B]_{ij} \).

(c) If \( A \in \tau_j \), then \([A]_i = [A]_{ij} \).

Proposition 1.12—(a) If \( \mathcal{F} \) is a filter on \( X \), then \( \theta_{ij} \)-lim \( \mathcal{F} \subset \theta_{ij} \)-adh \( \mathcal{F} \), \( \tau \)-lim \( \mathcal{F} \subset \theta_{ij} \)-lim \( \mathcal{F} \) and \( \tau \)-adh \( \mathcal{F} \subset \theta_{ij} \)-adh \( \mathcal{F} \).

(b) If \( \mathcal{F}, \mathcal{F}^* \) are two filters on \( X \) such that \( \mathcal{F} \subset \mathcal{F}^* \), then \( \theta_{ij} \)-lim \( \mathcal{F} \subset \theta_{ij} \)-lim \( \mathcal{F}^* \) and \( \theta_{ij} \)-adh \( \mathcal{F}^* \subset \theta_{ij} \)-adh \( \mathcal{F} \).

(c) If \( \mathcal{F} \) is a \( \tau \)-open filter on \( X \), the \( \theta_{ij} \)-adh \( \mathcal{F} = \tau \)-adh \( \mathcal{F} \).

Finally, we prove two useful lemmas.

Lemma 1.13—If \( (X, \tau_1, \tau_2) \) is \( (i, j) \)-semiregular, then \( \theta_{ij} \)-lim \( \mathcal{B} = \tau \)-lim \( \mathcal{B} \) for each \( \tau \)-open filterbase \( \mathcal{B} \) on \( X \).

Proof: By Proposition 1.12 \((x)\), \( \tau \)-lim \( \mathcal{B} \subset \theta_{ij} \)-lim \( \mathcal{B} \).

Let now \( x \) be a \( \theta_{ij} \)-limit point of \( \mathcal{B} \) and \( V \) a \( \tau \)-open neighbourhood of \( x \). Since \( (X, \tau_1, \tau_2) \) is \( (i, j) \)-semiregular there exists an \( (i, j) \)-regularly open neighbourhood \( W \) of \( x \) such that \( x \in W \subset V \). By the hypothesis \( x \in \theta_{ij} \)-lim \( \mathcal{B} \), so there exists a \( B \in \mathcal{B} \) with \( B \subset [W]_j \). Finally, since \( B \in \tau_1 \) and \( W = <[W]_j> \), it is clear that \( B \subset W \subset V \). Thus \( x \in \tau \)-lim \( \mathcal{B} \) and the proof is complete.

Lemma 1.14—If \( (X, \tau_1, \tau_2) \) is a bitopological space, then for each filter \( \mathcal{F} \) on \( X \) there exists a \( \tau \)-open filter \( \mathcal{F}^* \) on \( X \), coarser then \( \mathcal{F} \), such that \( \tau \)-adh \( \mathcal{F}^* = \theta_{ij} \)-adh \( \mathcal{F} \).

Proof: Let \( \mathcal{F} \) be a filter on \( X \). If \( \theta \)-adh \( \mathcal{F} = X \), then the \( \tau \)-open filter \( \mathcal{F}^* = \{X\} \) is coarser than \( \mathcal{F} \) and \( \tau \)-adh \( \mathcal{F}^* = X \). If \( \theta \)-adh \( \mathcal{F} \neq X \), then for
each \( x \in X - (\theta_{ij} \text{-adh } \mathcal{F}) \) there exist a \( V_x \in \tau_i(x) \) and an \( F_x \in \mathcal{F} \) such that \( F_x \cap [V_x]_j = \phi \). If now \( \mathcal{V} \) is the collection of all these \( V_x \)'s it is easily proved that \( \mathcal{B} = \{ X - [V]_j : V \in \mathcal{V} \} \) is a \( \tau_j \)-open filterbase which generates a filter \( \mathcal{F}^* \) on \( X \), coarser than \( \mathcal{F} \). Since \( \mathcal{V} \) is a \( \tau_i \)-open cover of \( X - (\theta_{ij} \text{-adh } \mathcal{F}) \), it is clear that

\[
\tau_i \text{-adh } \mathcal{F}^* = \bigcap \{ [X - [V]_j]_l : V \in \mathcal{V} \} = X = \bigcup \{ [\langle [V]_j \rangle]_i : V \in \mathcal{V} \} \subset \theta_{ij} \text{-adh } \mathcal{F}.
\]

Finally, by Proposition 1.12, \( \theta_{ij} \text{-adh } \mathcal{F} \subset \tau_i \text{-adh } \mathcal{F}^* \) and hence the proof is complete.

2. Pairwise Almost Compactness

Definition 2.1—A bitopological space \((X, \tau_1, \tau_2)\) is called \((i, j)\)-almost compact (henceforth abbreviated as \((i, j) - a - c\)) if given a point \( c \in X \), a \( \tau_i \)-open cover \( \mathcal{S} = \{ G_k : k \in K \} \) of \( X - \{c\} \) and a \( \tau_j \)-open neighbourhood \( V \) of \( c \), there exists a finite subcollection \( \{ G_{k_m} : m = 1, 2, \ldots, n \} \) of \( \mathcal{S} \) with \( X = [V]_j \cup (\bigcup \{ [G_{k_m}]_j : m = 1, 2, \ldots, n \}) \). If \((X, \tau_1, \tau_2)\) is \((1, 2)\)-and \((2, 1) - a - c\), then it is called pairwise almost compact (henceforth abbreviated as \( p - a - c \)).

The following result is an immediate consequence of the above definition and Lemma 1.5.

Theorem 2.2—If \((X, \tau_1, \tau_2)\) is \( p - a - c \), then the \( p \)-semiregularization \((X, \tau^p_1, \tau^p_2)\) of \((X, \tau_1, \tau_2)\) is also \( p - a - c \).

It is clear from the definitions and Example 2.3 below that \( p \)-a-compactness is strictly weaker than semi-compactness and \( p \)-compactness (Cooke and Reilly\(^8\), Swart\(^11\) and Fletcher et al.\(^4\))

Example 2.3—Let \( X = [0, 1] \). The collections

\[
\mathcal{B}_1 = \{X\} \cup \{(a, b) : 0 \leq a < b \leq 1\} \cup \{(a, 1) : 0 \leq a < 1\}
\]

\[
\cup \left\{ \left[0, b\right) - \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} : 0 < b \leq 1 \right\}
\]

\[
\mathcal{B}_2 = \{X\} \cup \{(a, b) : 0 \leq a < b \leq 1\} \cup \{(0, b) : 0 < b \leq 1\}
\]

\[
\cup \left\{(a, 1) - \left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\} : 0 \leq a < 1 \right\}
\]

are bases for the topologies \( \tau_1 \) and \( \tau_2 \) respectively.

The collection

\[
\mathcal{A} = \left\{ [0, 1) - \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}, \left( \frac{1}{2}, 1 \right) \right\} \cup \left\{ \left( \frac{1}{n + 1}, \frac{1}{n} \right) : n \in \mathbb{N} - \{1\} \right\}
\]

are bases for the topologies \( \tau_1 \) and \( \tau_2 \) respectively.
is clearly a $\tau_1, \tau_2$-open and also a pairwise open cover of $X$. Since any finite sub-
collection of $\mathcal{A}$ does not cover $X$, $(X, \tau_1, \tau_2)$ is neither semi-compact nor $p$-compact.
However, since the topological space $(X, \tau)$, where $\tau$ is the usual topology on $X$, is
compact and $[B]_j = [B]$, for each $B \in \mathcal{B}_1$, it is easily proved that $(X, \tau_1, \tau_2)$ is
$p - a - c$.

By the following examples we show that $B$-compactness (Cooke and Reilly$^3$ and
Birsan$^2$) does not imply and is not implied by $p$-a-compactness.

**Example 2.4**—Let $X = [0, 1]$, $\tau_1 = \{\phi, X, \{0\}\} \cup \{\{0, \alpha\} : 0 < \alpha \leq 1\}$ and $\tau_2 = \{\phi, X, \{1\}\} \cup \{\{a, 1\} : 0 \leq a < 1\}$. The bitopological space $(X, \tau_1, \tau_2)$ is clearly $B$-compact.
However, since there exist a $\tau_1$-open cover $\mathcal{A} = \left\{\left[0, 1 - \frac{1}{n}\right] : n \in \mathbb{N} - \{1\}\right\}$ of
$X - \{1\}$ and a $\tau_2$-open neighbourhood $V = \{1\}$ of the point $c = 1$ such that
for each finite subset $\{m_1, m_2, \ldots, m_n\}$ of $\mathbb{N} - \{1\}$, $[V]_1 \cup \bigcup_{k=1}^n \left\{\left[0, 1 - \frac{1}{m_k}\right] : k = 1, 2, \ldots, n\right\} = \{1\} \cup \left[0, 1 - \frac{1}{m_0}\right] \not\subseteq X$ $(m_0 = \max \{m_1, m_2, \ldots, m_n\})$, $(X, \tau_1, \tau_2)$ is not $p - a - c$.

**Example 2.5**—Let $(X, \tau_1, \tau_2)$ be the $p - a - c$ space of Example 2.3. Since the
topological spaces $(X, \tau_1)$ and $(X, \tau_2)$ are not compact, it is clear that $(X, \tau_1, \tau_2)$ is not
$B$-compact.

Finally, by the next two examples, we show that $p$-a-compactness does not
imply and is not implied by Mukherjee’s$^7$ $p$-a-compactness (henceforth denoted by
$M$-a-compactness).

**Example 2.6**—The topological space $(X, \tau_1, \tau_2)$, where $X = \mathbb{R}$, $\tau_1$ the discrete
topology and $\tau_2$ the topology of finite complements on $\mathbb{R}$, is clearly $p - a - c$ but it is not $M - a - c$.

**Example 2.7**—The bitopological space $(X, \tau_1, \tau_2)$ of Example 2.4 is not $p - a - c$.
However, since $(X, \tau_1)$ and $(X, \tau_2)$ are compact topological spaces, $(X, \tau_1, \tau_2)$ is
$M - a - c$.

### 3. Filter Characterizations of $p$-a-Compactness

**Theorem 3.1**—Let $(X, \tau_1, \tau_2)$ be a bitopological space. The following are equivalent.

(i) $(X, \tau_1, \tau_2)$ is $(i, j) - a - c$.

(ii) For each point $c \in X$ and for each $\tau_i$-open filterbase $\mathcal{B}$ on $X$ with
$\tau_i$-adh $\mathcal{B} \subseteq \{c\}$, $c \in \theta_i$-$\lim \mathcal{B}$.

(iii) For each point $c \in X$ and for each filter $\mathcal{F}$ on $X$ with $\theta_i$-$\text{adh} \mathcal{F} \subseteq \{c\}$,
$c \in \theta_i$-$\text{lim} \mathcal{F}$.
PROOF: (i) \Rightarrow (ii) Let \( c \) be a point of \( X \), \( \mathcal{B} \) a \( \tau_f \)-open filterbase on \( X \) with 
\( \tau_f \)-adh \( \mathcal{B} \subset \{c\} \) and \( V \) a \( \tau_f \)-open neighbourhood of \( c \). Since \( \bigcap \{[B]_i : B \in \mathcal{B} \} \subset \{c\} \), the collection \( \{X - [B]_i : B \in \mathcal{B} \} \) is a \( \tau_f \)-open cover of \( X - \{c\} \). Therefore there exists a finite subcollection \( \{B_k : k = 1, 2, \ldots, n\} \) of \( \mathcal{B} \) such that 
\[
X = [V]_i \cup \left( U \{[X - [B_k]_i]_j : k = 1, 2, \ldots, n\} \right) \]
\[
= [V]_i \cup (X - \bigcap \{[B_k]_j : k = 1, 2, \ldots, n\}).
\]

Finally, if we choose a \( B \in \mathcal{B} \) with \( B \subset B_1 \cap B_2 \cap \ldots \cap B_n \) it is clear that \( B \subset [V]_i \) and hence \( c \in \theta_{\mu \text{-lim}} \mathcal{B} \).

(ii) \Rightarrow (i) Suppose \((X, \tau_1, \tau_2)\) is not \((i, j) - a - c\). Then there exist a point 
\( c \in X \), a \( \tau_1 \)-open cover \( \{G_l : l \in L\} \) of \( X - \{c\} \) and a \( \tau_f \)-open 
neighbourhood \( V \) of \( c \), such that for each finite subset \( K \) of \( L \), \( X \neq [V]_i \cup \left( U \{[G_k]_l : k \in K\} \right) \).

Since now \( (X - [V]_i) \cap (\bigcap \{X - [G_k]_j\} : k \in K) \neq \emptyset \), it is easily proved that 
\( \mathcal{B} = \{ \bigcap \{X - [G_k]_j\} : K \subset L, K \text{ finite} \} \) is a \( \tau_f \)-open filterbase on \( X \) with 
\( \tau_f \)-adh \( \mathcal{B} \subset \{c\} \) such that \( B \notin [V]_i \) for each \( B \in \mathcal{B} \). Therefore \( c \notin \theta_{\mu \text{-lim}} \mathcal{B} \) and 
the proof is complete.

(ii) \Rightarrow (iii) Follows from Lemma 1.14 and Proposition 1.12 (b).

(iii) \Rightarrow (ii) Follows from Proposition 1.12 (c).

Theorem 3.2—If \((X, \tau_1, \tau_2)\) is \( p \)-Hausdorff, then the following are equivalent.

(i) \((X, \tau_1, \tau_2)\) is \((i, j) - a - c\).

(ii) For each \( \tau_f \)-open filterbase \( \mathcal{B} \) on \( X \) with \( \tau_l \)-adh \( \mathcal{B} = \{c\} \), \( c \in \theta_{\mu \text{-lim}} \mathcal{B} \).

(iii) For each filter \( \mathcal{F} \) on \( X \) with \( \theta_{\mu \text{-adh}} \mathcal{F} = \{c\} \), \( c \in \theta_{\mu \text{-lim}} \mathcal{F} \).

PROOF: (i) \Rightarrow (ii) Follows from Theorem 3.1.

(ii) \Rightarrow (i) We need only prove that for each \( \tau_f \)-open filterbase \( \mathcal{B} \) on \( X \) with 
\( \tau_l \)-adh \( \mathcal{B} = \emptyset \), \( \theta_{\mu \text{-lim}} \mathcal{B} = X \). Let \( \mathcal{B} \) be a \( \tau_f \)-open filterbase on \( X \) with \( \tau_l \)-adh \( \mathcal{B} = \emptyset \) 
and \( c \) a point of \( X \). It is easily proved that \( \mathcal{B}^* = \{B \cup G : B \in \mathcal{B} \text{ and } G \in \tau_f (c)\} \) 
is a \( \tau_f \)-open filterbase on \( X \) with \( \tau_l \)-adh \( \mathcal{B}^* = \{c\} \). So, by the hypothesis, \( c \in \theta_{\mu \text{-lim}} \mathcal{B}^* \). 
Since now the filter \( \mathcal{F}^* \) generated by \( \mathcal{F}^* \) is coarser than the filter \( \mathcal{F} \) generated 
by \( \mathcal{B} \), it is clear, by Proposition 1.12 (b), that \( c \in \theta_{\mu \text{-lim}} \mathcal{B} \) and the proof is 
complete.

(ii) \Rightarrow (iii) Follows from Lemma 1.14 and Proposition 1.12.

By the following example one can see that the condition "\( p \)-Hausdorff" can not 
be omitted in Theorem 3.2.
Example 3.3—Let \((\mathbb{R}^2, \tau_1, \tau_2)\) be the bitopological product \([\text{Swart}^{11}]\), of the spaces \((\mathbb{R}, \tau_1', \tau_2')\) and \((\mathbb{R}, \tau_1', \tau_2')\), where \(\tau_1'\) is the discrete topology, \(\tau_2' = \tau_1'\) the usual topology and \(\tau_2'\) the topology of finite complements on \(\mathbb{R}\). Clearly \((\mathbb{R}^2, \tau_1, \tau_2)\) is not \(p\)-Hausdorff. Since there exist a \(\tau_1\)-open cover \(\mathscr{A} = \{\{x\} \times \mathbb{R} : x \in \mathbb{R}\}\) of \(\mathbb{R}^2 - ((0,0))\) and a \(\tau_2\)-open neighbourhood \(V = (-1, 1) \times \mathbb{R}\) of the point \(c = (0,0)\) such that for each finite subset \(\{x_1, x_2, ..., x_n\}\) of \(\mathbb{R}\)

\[ [V]_1 \cup (\cup \{\{x_k\} \times \mathbb{R}\}_2 : k = 1, 2, ..., n) = ((-1,1) \cup \{x_1, x_2, ..., x_n\}) \times \mathbb{R} \neq \mathbb{R}^2 \]

\((\mathbb{R}^2, \tau_1, \tau_2)\) is not \((1, 2) - a - c\).

However, we can easily prove that property (ii) (and also the equivalent property (iii)) of Theorem 3.2 does hold in \((\mathbb{R}^2, \tau_1, \tau_2)\). In fact if \(\mathscr{B}\) is a \(\tau_2\)-open filterbase on \(\mathbb{R}^2\) and \((x_0, y_0) \in \tau_1 - \text{adh } \mathscr{B}\) it is easy to see that for each \(y \in \mathbb{R}\), \((x_0, y)\) is also a \(\tau_1\)-adherent point of \(\mathscr{B}\). That means that each \(\tau_2\)-open filterbase on \(\mathbb{R}^2\) with non-empty \(\tau_1\)-adherence has an infinite number of \(\tau_1\)-adherent points and hence \((\mathbb{R}^2, \tau_1, \tau_2)\) has property (ii) (and also property (iii)).

4. Minimal \(p\)-Hausdorff Spaces

The concept of minimal \(p\)-Hausdorff bitopological spaces was initiated by Raghavan and Reilly\(^8\) as follows.

Definition 4.1—A bitopological space \((X, \tau_1, \tau_2)\) is called minimal \(p\)-Hausdorff if it is \(p\)-Hausdorff and if \((X, \tau_3, \tau_4)\) is \(p\)-Hausdorff with \(\tau_3 \subseteq \tau_1\) and \(\tau_4 \subseteq \tau_2\), then \(\tau_1 = \tau_3\) and \(\tau_2 = \tau_4\).

Raghavan and Reilly\(^8\) gave the following characterization of minimal \(p\)-Hausdorff spaces.

Theorem 4.2\(^8\)—If \((X, \tau_1, \tau_2)\) is \(p\)-Hausdorff, then the following are equivalent.

(a) \((X, \tau_1, \tau_2)\) is minimal \(p\)-Hausdorff.

(b) For each \(\tau_1\)-open filterbase \(\mathscr{B}_1\) and for each \(\tau_2\)-open filterbase \(\mathscr{B}_2\) on \(X\) with \(\tau_2\text{-adh } \mathscr{B}_1 = \tau_1\text{-adh } \mathscr{B}_2 = \{p\}\), \(\mathscr{B}_1\) is \(\tau_1\)-convergent to \(p\) and \(\mathscr{B}_2\) is \(\tau_2\)-convergent to \(p\).

By the following result we show that there is no need of the sharing of the point \(p\) in the above Theorem.

Theorem 4.3—If \((X, \tau_1, \tau_2)\) is \(p\)-Hausdorff, then the following are equivalent.

(i) \((X, \tau_1, \tau_2)\) is minimal \(p\)-Hausdorff

(ii) Each \(\tau_i\)-open filterbase \(\mathscr{B}_i\) on \(X\) with a unique \(\tau_j\)-adherent point \(x_i\) is \(\tau_i\)-convergent to \(x_i\) \((i, j = 1, 2, i \neq j)\).
Proof: We need only to prove that (b) of Theorem 4.2 implies (ii) of Theorem 4.3. Let \( \mathcal{B}_i \) be a \( \tau_i \)-open filterbase on \( X \) with \( \tau_j \)-adh \( \mathcal{B}_i = \{x_i\} \). Since \( (X, \tau_1, \tau_2) \) is \( p \)-Hausdorff, \( \tau_i \)-adh \( \tau_j(x_i) \) = \( \cap \{\mathcal{V} : \mathcal{V} \in \tau_j(x_i)\} = \{x_i\} \) (Reilly\(^4\)). Therefore, by (b) of Theorem 4.2, \( x_i \in \tau_i \)-lim \( \mathcal{B}_i \).

Our next result is a generalization of the following well-known Theorem 1.4 of Katetov\(^5\): A Hausdorff topological space is minimal Hausdorff iff it is almost compact and semiregular.

**Theorem 4.4**—A \( p \)-Hausdorff space \( (X, \tau_1, \tau_2) \) is minimal \( p \)-Hausdorff iff it is \( p - a - c \) and \( p \)-semiregular.

Proof: Let \( (X, \tau_1, \tau_2) \) be a minimal \( p \)-Hausdorff bitopological space and \( \mathcal{B}_i \) a \( \tau_i \)-open filterbase on \( X \) with \( \tau_j \)-adh \( \mathcal{B}_i = \{x_i\} \) \( i, j = 1, 2, i \neq j \). By Theorem 4.3 and Proposition 1.12, \( \mathcal{B}_i \) is \( \theta_i \)-convergent to \( x_i \). Thus, by Theorem 3.2, \( (X, \tau_1, \tau_2) \) is \( p - a - c \).

By Theorem 1.8, the \( p \)-semiregularization \( (X, \tau_1^*, \tau_2^*) \) of \( (X, \tau_1, \tau_2) \) is \( p \)-Hausdorff. Since now \( (X, \tau_1, \tau_2) \) is minimal \( p \)-Hausdorff, \( \tau_1^* = \tau_1 \) and \( \tau_2^* = \tau_2 \). Therefore, by Theorem 1.4, \( (X, \tau_1, \tau_2) \) is \( p \)-semiregular.

Conversely, let \( (X, \tau_1, \tau_2) \) be a \( p \)-Hausdorff \( p - a - c \) and \( p \)-semiregular bitopological space and \( \mathcal{B}_i \) a \( \tau_i \)-open filterbase on \( X \) with \( \tau_j \)-adh \( \mathcal{B}_i = \{x_i\} \) \( i, j = 1, 2, i \neq j \). By Theorem 3.2 and Lemma 1.13, \( x_i \in \tau_i \)-lim \( \mathcal{B}_i \). Thus, by Theorem 4.3, \( (X, \tau_1, \tau_2) \) is minimal \( p \)-Hausdorff.

The following result is an immediate consequence of Theorems 1.7, 1.8, Proposition 2.2 and Theorem 4.4.

**Theorem 4.5**—If \( (X, \tau_1, \tau_2) \) is \( p \)-Hausdorff and \( p - a - c \), then the \( p \)-semiregularization \( (X, \tau_1^*, \tau_2^*) \) of \( (X, \tau_1, \tau_2) \) is minimal \( p \)-Hausdorff.

Another well-known characterization of minimal Hausdorff topological spaces is the following: A Hausdorff topological space is minimal Hausdorff iff it is minimal semiregular Hausdorff. We note that necessity is obvious by Katetov's Theorem and that sufficiency is proved by Banaschewski\(^1\) (p. 147).

Theorem 4.7 below is a bitopological analogue of the above characterization.

**Definition 4.6**—A bitopological space \( (X, \tau_1, \tau_2) \) is called minimal \( p \)-semiregular \( p \)-Hausdorff if it is \( p \)-semiregular and \( p \)-Hausdorff and if \( (X, \tau_3, \tau_4) \) is \( p \)-semiregular and \( p \)-Hausdorff with \( \tau_3 \subseteq \tau_1 \) and \( \tau_4 \subseteq \tau_2 \), then \( \tau_1 = \tau_3 \) and \( \tau_2 = \tau_4 \).

**Theorem 4.7**—A bitopological space \( (X, \tau_1, \tau_2) \) is minimal \( p \)-Hausdorff iff it is minimal \( p \)-semiregular \( p \)-Hausdorff.
PROOF: Necessity follows from Theorem 4.4. To show sufficiency assume that 
\((X, \tau_1, \tau_2)\) is a minimal \(p\)-semiregular \(p\)-Hausdorff space which is not minimal \(p\)-Hausdorf. Then there exist two topologies \(\tau_3 \subset \tau_1\) and \(\tau_4 \subset \tau_2\) with \(\tau_3 \neq \tau_1\) or \(\tau_4 \neq \tau_2\) \(p\)-Hausdorff. By Theorems 1.7 and 1.8, the \(p\)-semiregularization \((X, \tau_3^*, \tau_4^*)\) of \((X, \tau_3, \tau_4)\) is \(p\)-semiregular and \(p\)-Hausdorff. Since now \(\tau_3^* \subset \tau_1\), \(\tau_4^* \subset \tau_2\) and \(\tau_3^* \neq \tau_1\) or \(\tau_4^* \neq \tau_2\), \((X, \tau_3^*, \tau_4^*)\) is not minimal \(p\)-semiregular \(p\)-Hausdorff and the contradiction completes the proof.

The following result follows immediately from the definitions and from the fact that \(p\)-Hausдорffness is a projective and productive property.\(^{11}\)

Theorem 4.8—If the non empty bitopological product \((X, \tau_1, \tau_2)\) of the family \(((X_k, \tau_{1k}, \tau_{2k}))_{k \in K}\) [Swart\(^{11}\)] is minimal \(p\)-Hausdorff, then each coordinate space \((X_k, \tau_{1k}, \tau_{2k})\) is minimal \(p\)-Hausdorff.

Finally, by the following example it is shown that the converse of Theorem 4.8 does not hold. So, the concept of minimal \(p\)-Hausdorff spaces is not product invariant.

Example 4.9—Let \(\tau_1^*\) be the discrete topology on \(R\), \(\tau_2^*\) the topology of finite complements on \(R\), \(\tau_1^* = \tau_2^*\) and \(\tau_2^* = \tau_1^*\). It is known\(^6\) that the bitopological spaces \((R, \tau_1^*, \tau_2^*)\) and \((R, \tau_1^*, \tau_2^*)\) are minimal \(p\)-Hausdorff. Let now \((R^2, \tau_1, \tau_2)\) be the bitopological product of the above spaces. The classes \(\mathcal{B}_1 = \{\{x\} \times A : x \in R\) and \(\mathcal{B}_2 = \{A \times \{x\} : x \in R\) are clearly bases for the topologies \(\tau_1\) and \(\tau_2\) respectively. If \(p = (p_1, p_2)\) is a point of \(R^2\) it is easily proved that the classes \(\mathcal{B}_1^* = \{A : A \in \mathcal{B}_1\) and \(p \in A\}) \cup \{(R \times B) \cup C : R \subset B\) finite, \(C \in \mathcal{B}_1\) and \(p \in C\), \(\mathcal{B}_2^* = \{A : A \in \mathcal{B}_2\) and \(p \in A\}) \cup \{(B \times R) \cup C : R \subset B\) finite, \(C \in \mathcal{B}_2\) and \(p \in C\) are bases for the topologies \(\tau_1^*\) and \(\tau_2^*\) on \(R^2\), which are strictly weaker than \(\tau_1\) and \(\tau_2\) respectively. Since now \((R^2, \tau_1^*, \tau_2^*)\) is \(p\)-Hausdorff it is obvious that the \(p\)-Hausdorff space \((R^2, \tau_1, \tau_2)\) is not minimal \(p\)-Hausdorff.

The same conclusion can be obtained immediately by Theorem 4.4. In fact it is easily proved that each \(B \in \mathcal{B}_1\) is \((1,2)\)-regularly open and each \(B \in \mathcal{B}_2\) \((2,1)\)-regularly open and hence \((R^2, \tau_1, \tau_2)\) is \(p\)-semiregular. However, since there exist a point \(c = (0, 0) \in R^2\), a \(\tau_1\)-open cover \(\mathcal{A} = \{\{x\} \times R : x \in R\} \) of \(R^2 - \{c\}\) and a \(\tau_2\)-open neighbourhood \(V = R \times \{0\}\) of \(c\), such that for each finite subset \(\{x_1, x_2, ..., x_n\}\) of \(R\), \([\mathcal{V}]_n \cup (\cup \{[x_k] \times R]_n : k = 1, 2, ..., n\} = (R \times \{0\}) \cup \{x_1, x_2, ..., x_n\} \times R \neq R^2\),
\((\mathbb{R}^2, \tau_1, \tau_2)\) is not \(p-a-c\). Therefore, by Theorem 4.4, \((\mathbb{R}^2, \tau_1, \tau_2)\) is not minimal \(p\)-Hausdorff.

References