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DETERMINISTIC LOT SIZE MODEL WITH *n*TH POWER TIME PROPORTIONAL DEMAND

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Classical lot-size model with demand rate deterministic and increasing with time has been developed. The optimal schedule of replenishments is derived. The technique which determines, the unique optimal value of replenishment and the unique optimal scheduling of replenishment has been developed.

1. INTRODUCTION

In this paper an inventory system for a classical lot-size model with deterministic demand rate increasing with time has been developed. The demand considered is proportional to the *n*th power of the time i. e.

$$r(t) = a\,t^n,\ a > 0,\ 0 < n \leqslant 1 \text{ and } r = 0 \text{ when } t = 0$$

whereas in Resh *et al.*[3] $r(t) = at$, $a > 0$.

The problem is to find the optimal schedule of replenishments so as to minimize the total replenishment cost and inventory carrying cost throughout the planning horizon.

2. FORMULATION AND SOLUTION

As in Resh *et al.*[2], the general expression for the total cost function C is

$$C = I.\,I_c + m.\,A \qquad \ldots(2.1)$$

where I is the total inventory carried throughout T; m the number of replenishments T; I_c the inventory carrying cost per unit, per unit time; and A the replenishment cost per order.

The value of I is derived below.

Let q_i be the lot-size quantity of the ith replenishment period elapsing from t_i until t_{i+1}, where $i = 0, 1, 2, \ldots, m-1$

$$q_i = \int_{t_i}^{t_{i+1}} r(t)\,dt = a\left[\, t_{i+1}^{n+1} - t_i^{n+1} \right] / (n+1). \qquad \ldots(2.2)$$

Let $I_i\ (t)$ be the stock level at t, $t_i \leqslant t \leqslant t_{i+1}$

$$I_i(t) = q_i - \int_{t_i}^{t} r(u)\,du$$

$$= a\left[\, t_{i+1}^{n+1} - t^{n+1} \,\right] / (n+1). \qquad \ldots(2.3)$$

Now I_i denote the total inventory carried throughout the ith period

$$I_i = \int_{t_i}^{t_{i+1}} I_i(t)\,dt$$

$$= a\left[\, \frac{t_{i+1}^{n+2}}{n+2} + \frac{t_i^{n+2}}{(n+1)(n+2)} - \frac{t_i\ t_{i+1}^{n+1}}{(n+1)} \,\right]. \qquad \ldots(2.4)$$

Therefore, for an m-replenishment system the total inventory is

$$I^{(m)}(\mathbf{t}) = \sum_{i=0}^{m-1} I_i\,, \text{ for } m = 2, 3, \ldots.$$

i. e.

$$I^{(m)}(\mathbf{t}) = \frac{a}{n+2}\,T^{n+2} - \frac{a}{n+1}\sum_{i=1}^{m-1} t_i\left[\, t_{i+1}^{n+1} - t_i^{n+1} \,\right]$$

and

$$I^{(1)} = \left(\frac{a}{n+2}\right) T^{n+2}. \qquad \ldots(2.5)$$

Now as in Resh *et al.*[2], we find an integer m and a vector of m components $\mathbf{t}^{(m)} = (t_0, t_1, \ldots, t_{m-1})$ where $t_0 = 0$, $t_i \leqslant t_{i+1} \leqslant T$, so as to minimize

$$C^{(m)}(\mathbf{t}) = I^{(m)}.\ I_c + m.\ A. \qquad \ldots(2.6)$$

We follow the approach of Resh *et al.*[2], i. e. the two-stage optimization procedure :

(1) For a fixed value of m, find a vector $\mathbf{t}^{*(m)}$ that minimizes $I^{(m)}(\mathbf{t})$.

(2) $I^{*(m)}$ is a decreasing and convex function of m, for the first m that satisfies $I^{*(m*)} - I^{*(m*-1)} \leqslant A/I_c$

where m^* be the optimal value of m and $I^{*(m)}$ be the value of $I^{(m)}(\mathbf{t})$ at the point $\mathbf{t}^{*(m)}$, *For stage* (1)

Let

$$\bar{I}^{(m)}(\mathbf{t}) = \sum_{i=1}^{m-1} t_i\left[\, t_{i+1}^{n+1} - t_i^{n+1} \,\right] \qquad \ldots(2.7)$$

and $\bar{I}^{(1)}(t) = 0$. Now the vector t which maximizes the eqn. (2.7), minimizes eqn. (2.5) for $m = 2, 3, \ldots$.

Lemma 1—Let $Q^{(m)}$ be a sequence such that $Q^{(1)} = 0$ and

$$Q^{(m)} = K(1 - Q^{(m-1)})^{-\frac{1}{n+1}} \text{ for } m \geqslant 2 \qquad \ldots(2.8)$$

where

$$K = (n+1)\left(\frac{1}{n+2}\right)^{(n+2)/(n+1)} \text{ then,}$$

(a) $Q^{(m)}$ is a monotonically increasing sequence;

(b) $Q^{(m)} \leqslant \left(\frac{n+1}{n+2}\right)$ for every integer m;

(c) $\lim_{m\to\infty} Q^{(m)} = \frac{n+1}{n+2}$.

PROOF : This can be proved adopting the line of argument followed in Lemma 2 of Resh *et al.*[2] by taking $f(x) = K(1-x)^{-\frac{1}{n+1}}$ where $0 \leqslant x \leqslant 1$ and show that the line $y = x$ is tangent to the above $f(x)$ at $x = \frac{n+1}{n+2}$.

Lemma 2—Let $g^{(m)}(T) = \underset{t}{\text{Max.}}[\bar{I}^{(m)}(t)]$ where $0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_{m-1} \leqslant T$ satisfy

$$g^{(m)}\ T = Q^{(m)}.\ T^{n+2} \text{ for } m \geqslant 1 \qquad \ldots(2.9)$$

where $Q^{(m)}$ is given by Lemma 1.

PROOF : Employing Bellman's Principle of optimality and the method of induction we obtain

$$g^{(m)}(T) = \underset{t\leqslant T}{\text{Max}}\ [g^{(m-1)}(t) + t\,(T^{n+1} - t^{n+1})] \qquad \ldots(2.10)$$

since

$$g^{(1)}(T) = 0, \quad g^{(2)}(T) = \underset{t\leqslant T}{\text{Max}}\ [t\,(T^{n+1} - t^{n+1})].$$

The maximum point is $t = \left(\frac{1}{n+2}\right)^{(n+2)/(n+1)}$

and

$$g^{(2)}(T) = (n+1)\left(\frac{1}{n+2}\right)^{(n+2)/(n+1)}.\ T^{n+2} = Q^{(2)}.\ T^{n+2}.$$

Therefore using (2.9) in (2.10), we get

$$g^{(m)}(T) = \underset{t \leqslant T}{\text{Max}} \left[Q^{(m-1)}.\ t^{n+2} + t\,(T^{n+1} - t^{n+1})\right] \qquad \ldots(2.11)$$

where maximum is achieved at $t = T[(n+2)(1-(Q^{(m-1)})]^{-1/n+1} < T$ (by Lemma 1). Using the value of t from eqn. (2.8) and (2.9) in eqn. (2.11), the value of $I^{*(m)}$ becomes

$$I^{*(m)} = \frac{a}{n+2} T^{n+2} - \frac{a}{n+1} Q^{(m)} T^{n+2}. \qquad \ldots(2.12)$$

Lemma 3— $\Delta Q^{(m)}$ strictly decreases as m increases.

PROOF : We have (i) $f(x) - x$ is strictly decreasing over the interval $[0, (n+1)(n+2)]$ and (ii) $Q^{(m)}$ is a strictly increasing sequence bounded by $(n+1)/(n+2)$.

Therefore

$$\Delta Q^{(m)} = Q^{(m)} - Q^{(m-1)} = K(1 - Q^{(m-1)})^{-\frac{1}{n+}} - Q^{(m-1)}$$

strictly decreases as m increases.

Hence by Lemmas 1 and 3, $I^{*(m)}$ is a monotonically decreasing sequence bounded by zero and $\underset{m \to \infty}{\text{limit}}\ I^{*(m)} = 0$.

Now $t_{m-1}^{*(m)}$ can be written in term of T as $t_{m-1}^{*(m)} = \alpha_{m-1} T$ where

$$\alpha_{m-1} = [(n+2)(1 - Q^{(m-1)})]^{-1/n+1} = \left(\frac{n+2}{n+1}\right) Q^{(m)}, \text{ for } m = 1, 2, \ldots.$$

using eqn. (2.8).

$$\alpha_m = [(n+2) - (n+1)\,\alpha_{m-1}]^{-1/(n+1)} \text{ for } m = 1, 2, \ldots.$$

$\because \underset{m \to \infty}{\text{limit}}\ \alpha_m = 1$ since $Q^{(m)}$ is monotonically increasing and bounded by 1, a backwards substitution, starting at $t_m = T$, yields $t_i^{*(m)} = T \prod_{j=i}^{m-1} \alpha_j$.

Resh *et al.*[2] observe that $t_i^{*(m)}$ is independent of a and is unique and that $t_i^{*(m)} < t_{i+1}^{*(m)}$. We can now go to the second stage of the optimization procedure, to find the optimal value of m. Equation (2.5) contains only one variable namely m. The local minimum of $C^{*(m)} = \underset{t}{\text{Min}}\,[C^{(m)}(t)]$ is achieved at m^* which satisfies

$$I^{*(m*)} - I^{*}(m^* + 1) \leqslant \frac{A}{I_c} \leqslant I^{*(m*-1)} - I^{*(m*)}$$

by substituting the value of $I^{*(m*)}$, we get

$$\Delta\ Q^{(m*+1)} \leqslant \frac{(n+1)\,A}{a.\,(n+2)\,I_c} \leqslant \Delta\ Q^{(m*)} \qquad \ldots(2.13)$$

where

$$\Delta Q^{(m)} = Q^{(m)} - Q^{(m-1)}, \quad m = 2, 3, \ldots.$$

For establishing the uniqueness of m^*, which is determined by eqn. (2.13). We shall prove that

$$\Delta Q^{(m+1)} < \Delta Q^{(m)} \quad \text{for } m = 2, 3, \ldots.$$

Thus we prove on the similar lines as in Lemma 3 of Resh *et al.*[2] i. e.

$$\frac{\Delta Q^{(m+1)}}{\Delta Q^{(m)}} = \beta_m^{n+2}$$

where

$$\beta_m^{n+2} = \frac{(Q^{(m+1)})^{n+1} \cdot (Q^{(m)})^{n+1}}{K^{n+1}\left[Q^{(m+1)})^n + (Q^{(m+1)})^{n-1} \cdot Q^{(m)} + \ldots + Q^{(m+1)}(Q^{(m)})^{n-1} + Q^{(m)})^n\right]}.$$

By definition β_m^{n+2} is an increasing function for $Q^{(m)} > 0$ and $Q^{(m+1)} > 0$ since its derivative with respect to each of these variables is always positive, i. e.

$$\beta_m^{n+2} < \frac{(Q^{(m+1)})^{n+2}}{K^{n+1}(n+1)} = \left[\frac{n+2}{n+1} \cdot Q^{(m+1)}\right]^{n+2} = \alpha_m^{n+2}$$

and

$$\beta_m^{n+2} > -\frac{(Q^{(m)n+2}}{(n+1)K^{n+1}} = \left[\frac{n+2}{n+1} Q^{(m)}\right]^{n+2} = \alpha_{m-1}^{n+2}.$$

Therefore for every $m = 2, 3, \ldots$

$$\alpha_{m-1} < \beta_m < \alpha_m.$$

Using above relation, we get

$$\Delta Q^{(m+1)} = \beta_m^{n+2} \Delta Q^{(m)} < \alpha_m^{n+2} \Delta Q^{(m)} < \Delta Q^{(m)}$$

since $\alpha_m < 1$ for every m.

Hence m^* determined from eqn. (2.13) is unique and the minimal cost is $C^*(m^*)$. If $n = 1$, result of Resh *et al.*[2] is a particular case.

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