Indian J. pure appl. Math., **18** (1) : 27-31, January 1987

HEIGHT AND SEMIMODULARITY IN FINITE 0, 1-POSETS

JUHANI NIEMINEN

Department of Mathematics, Faculty of Technology, University of Oulu, 90570 *Oulu, Finland*

(*Received* 20 *May* 1985; *after revision* 29 *September* 1986)

A concept of semimodularity is defined in a finite 0, 1-poset so that $\mathbf{h}(a) + \mathbf{h}(b) \geqslant \mathbf{h}(a \vee b) + \mathbf{h}(a \wedge b)$ holds for any two elements $a, b \in \mathbf{P}_{01}$ in any χ-lattice derived from $\mathbf{P}_{01}$. The connection between modularity and semi-modularity is described.

1. INTRODUCTION AND BASIC CONCEPTS

A poset $\mathbf{P}$ with a least element 0 and a greatest element 1 is called an 0, 1-poset and denoted briefly by $\mathbf{P}_{01}$. We shall consider here finite posets only. Because $\mathbf{P}_{01}$ is finite, the set *mub* $\{a, b\}$ of minimal upper bounds of a and b as well as the set *mlb* $\{a, b\}$ of maximal lower bounds of a and b is nonempty for any two elements $a, b \in \mathbf{P}_{01}$. We associate to $\mathbf{P}_{01}$ an algebra, called a χ-lattice, by defining two binary operations join $\vee$ and meet $\wedge$ as follows: $a \vee b$ is a freely chosen element from *mub* $\{a, b\}$ and $a \wedge b$ also a freely chosen element from *mlb* $\{a, b\}$. After making the choice, the chosen element is fixed. The choice does not depend on the order of elements a and b, and thus $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$. Note that if *mub* $\{a, b\}$ = *mub* $\{c,d\}$ for $d \neq a, b$, then $a \vee b$ and $c \vee d$ need not be equal; an analogy holds for meets. The choice process is performed by means of a choice function χ, and thus we can also write $a \wedge b = \chi(mlb\ \{a, b\})$ and $a \vee b = \chi(mub\ \{a, b\})$. If $\mathbf{P}_{01}$ is not a lattice, one can associate to $\mathbf{P}_{01}$ several disjoint χ-lattices, each of which determine the order structure of $\mathbf{P}_{01}$ by the rule $a \leqslant b \Leftrightarrow a \wedge b = a \Leftrightarrow a \vee b = b$. Note that the join and meet of a χ-lattice are not associative operations. Properties of χ-lattices are given in the earlier works[2-4]. A χ-lattice L is modular if the laws M_1 and M_2 below hold for all $a, b, \in L$ (see Nieminen[3])

$$(M_1) \quad a \vee (b \wedge (a \vee c)) \geqslant (a \vee b) \wedge (a \vee c);$$

$$(M_2) \quad a \wedge (b \vee (a \wedge c)) \leqslant (a \wedge b) \vee (a \wedge c).$$

An 0,1-poset $\mathbf{P}_{01}$ is modular if every χ-lattice derived from $\mathbf{P}_{01}$ is modular.

A nonempty subset of $\mathbf{P}_{01}$ is called a subposet of $\mathbf{P}_{01}$ if $\mathbf{A} \cap$ *mub* $\{a, b\} \neq \phi \neq \mathbf{A} \cap$ *mlb* $\{a, b\}$ in $\mathbf{P}_{01}$ for any two elements $a, b \in \mathbf{A}$. The notation $a \parallel b$ means that the elements a and b are incomparable. An element a covers an element b, in notation, $a \succ b$, if $a > b$ and $a \geqslant c \geqslant b$ holds only when $a = c$ or $b = c$. A pentagon $\mathbf{P}_5$ is

an 0, 1-poset with elements, 0, 1, **a**, **b**, **c**, where $1 > \mathbf{a} > \mathbf{b} > 0$, $1 > \mathbf{c} > 0$ and $\mathbf{c} \| \mathbf{a}, \mathbf{b}$. $\mathbf{P}_5$ is the wellknown least nonmodular lattice. Clearly, if $\mathbf{P}_{01}$ is modular, it cannot contain $\mathbf{P}_5$ as a subposet.

2. Algebraic Semimodularity

In Birkhoff's book[1] (Chapt. II. 8) a poset is called semi-modular, if two elements **a** and **b**, $\mathbf{a} \neq \mathbf{b}$, cover an element **c**, then there exists an element $\mathbf{d} \succ \mathbf{a}, \mathbf{b}$. This property is rather weak and therefore we introduce a new semimodurity concept. We say that a finite 0, 1-poset $\mathbf{P}_{01}$ is 'algebraically semimodular' if the following two conditions hold for all $\mathbf{a}, \mathbf{b}, \mathbf{x} \in \mathbf{P}_{01}$ in any χ-lattice derived from $\mathbf{P}_{01}$:

(1) If $a \| b$ and $a \wedge b < x < a$, there is an element t with $a \wedge b < t \leqslant b$ and $(x \vee t) \wedge a = x$.

(2) If $\mathbf{a} \prec k$ for some $\mathbf{k} \in mub\ \{a, b\}$, then $\mathbf{a} \prec \mathbf{j}$ for each $\mathbf{j} \in mub\ \{a, b\}$.

Theorem 1—Let $\mathbf{P}_{01}$ be a finite algebraically semimodular 0, 1-poset. If $\mathbf{a} \neq \mathbf{b}$, and there is an element $\mathbf{c} \prec \mathbf{a}, \mathbf{b}$, then $\mathbf{a}, \mathbf{b} \prec \mathbf{k}$ for every $\mathbf{k} \in mub\ \{a, b\}$.

Proof : Let $\mathbf{c} \prec \mathbf{a}, \mathbf{b}$ and $\mathbf{k} \in mub\ \{a, b\}$. We consider the situation in a χ-lattice, where $\mathbf{a} \wedge \mathbf{b} = \mathbf{c}$ and $\mathbf{k} = \mathbf{a} \vee \mathbf{b}$. If **a** is not covered by $\mathbf{a} \vee \mathbf{b}$, there is an element **q** with $\mathbf{a} \prec \mathbf{q} \leqslant \mathbf{a} \vee \mathbf{b}$. Because $\mathbf{a} \wedge \mathbf{b} \prec \mathbf{b}$ and $\mathbf{a} \wedge \mathbf{b} \prec \mathbf{a} \prec \mathbf{q}$, $a \wedge b \in mlb\ \{q, b\}$ and we can choose $\mathbf{q} \wedge \mathbf{b} = \mathbf{a} \wedge \mathbf{b}$. Now (1) in the algebraic semimodularity and the relation $\mathbf{q} \wedge \mathbf{b} < \mathbf{a} < \mathbf{q}$ imply the existence of an element t such that $\mathbf{q} \wedge \mathbf{b} < \mathbf{t} \leqslant \mathbf{b}$ and $(\mathbf{a} \vee \mathbf{t}) \wedge \mathbf{q} = \mathbf{a}$. Because $\mathbf{q} \wedge \mathbf{b} = \mathbf{a} \wedge \mathbf{b} \prec \mathbf{b}$, $\mathbf{t} = \mathbf{b}$, and we obtain $(\mathbf{a} \vee \mathbf{b}) \wedge \mathbf{q} = \mathbf{q} = \mathbf{a}$ which is absurd. Hence $\mathbf{a} \wedge \mathbf{b} \prec \mathbf{a}, \mathbf{b}$ implies $\mathbf{a}, \mathbf{b} \prec \mathbf{a} \vee \mathbf{b} = \mathbf{k}$, and the theorem follows.

Before proving a connection between the modularity and algebraic semimodularity we need some auxiliary results.

Lemma 2—Let $\mathbf{P}_{01}$ be a finite modular 0, 1-poset, $\mathbf{x} \neq \mathbf{y}$, $\mathbf{a} \neq \mathbf{b}$, $\mathbf{x}, \mathbf{y} \in mlb\ \{a, b\}$ and $\mathbf{a}, \mathbf{b} \succ \mathbf{x}, \mathbf{y}$. Then for any $\mathbf{z} \in \mathbf{P}_{01}$ either $\mathbf{z} \geqslant \mathbf{k}$ for some $\mathbf{k} \in mub\{x, y\}$ or $\mathbf{z} \leqslant \mathbf{s}$ for some $\mathbf{s} \in mlb\ \{a, b\}$.

Proof : When $\mathbf{x}, \mathbf{y} \in mlb\ \{a,b\}$ and $\mathbf{a}, \mathbf{b} \succ \mathbf{x}, \mathbf{y}$, then clearly $\mathbf{a}, \mathbf{b} \in mub\{x, y\}$. Assume first that $\mathbf{z} \| \mathbf{k}$ for every $\mathbf{k} \in mub\ \{x, y\}$, and let $\mathbf{s} \in mlb\ \{a, b\}$. The relation $\mathbf{z} \leqslant \mathbf{s}$ cannot hold because then $\mathbf{z} \leqslant \mathbf{s} \leqslant \mathbf{a} \in mub\ \{x, y\}$ and $\mathbf{z} \| \mathbf{k}$ for any $\mathbf{k} \in mub\ \{x, y\}$. If $\mathbf{z} > \mathbf{s}$ there is an element **c** with $\mathbf{z} \geqslant \mathbf{c} \succ \mathbf{s}$. According to the dual of Theorem 5 of Nieminen[4], if $\mathbf{p} \succ \mathbf{d}$, $\mathbf{d} \in mlb\ \{e, f\}$, then $\mathbf{p} \succ \mathbf{j}$ for any $\mathbf{j} \in mlb\ \{e, f\}$, and in the present case this means that $\mathbf{c} \succ \mathbf{j}$ for any $\mathbf{j} \in mlb\ \{a,b\}$. Then $\mathbf{z} \geqslant \mathbf{c} > \mathbf{x}, \mathbf{y}$, and thus **z** is comparable to at least one minimal upper bound **k** of **x** and **y**; a contradiction. Hence $\mathbf{z} \| \mathbf{j}$ for any $\mathbf{j} \in mlb\ \{\mathbf{a}, \mathbf{b}\}$. But then there is at least one $\mathbf{m} \in mub\ \{y, z\}$ such that $y \prec \mathbf{k}_1 \prec \mathbf{m}$ with $\mathbf{k}_1 \in mub\ \{x, y\}$ and at least one **t** such that $\mathbf{t} \in mlb\ \{k_1, \mathbf{z}\}$ and $\mathbf{y} > \mathbf{t}$. Then the elements **y**, $\mathbf{k}_1$, **m**, **t** and **z** constitute a pentagon. The elements of this pentagon contradict in some χ-lattice **L** derived from $\mathbf{P}_{01}$ either

the rule M_1 or M_2. This is absurd, and thus the assumption $\mathbf{z} \parallel \mathbf{k}$ for any $\mathbf{k} \in mub\ \{x, y\}$ cannot hold. The proof for $\mathbf{z} \parallel \mathbf{s}$ for any $\mathbf{s} \in mlb\ \{a, b\}$ is dual. By combining these partial results we obtain the assertion of the lemma.

Lemma 3—Let $\mathbf{P}_{01}$ be a modular finite 0, 1-poset. If $\mathbf{a} \parallel \mathbf{b}$ in $\mathbf{P}_{01}$, then $\mathbf{A} = \{\mathbf{x} \mid \mathbf{x} \in \mathbf{P}_{01}$ and $\mathbf{a} \wedge \mathbf{b} \leqslant \mathbf{x} \leqslant \mathbf{a} \vee \mathbf{b}$ in some χ-lattice derived from $\mathbf{P}_{01}\}$ is a lattice as a poset.

PROOF : Let $\mathbf{c}, \mathbf{d} \in \mathbf{A}$. We shall consider the set $mub\ \{c, d\}$ only; the proof for $mlb\ \{c, d\}$ is dual. Because $\mathbf{a} \vee \mathbf{b} \geqslant \mathbf{c}, \mathbf{d}$, $mub\ \{\mathbf{c}, \mathbf{d}\} \cap \mathbf{A} = \mathbf{S} \neq \phi$. If S contains only one element, there is nothing to prove. Hence we assume that $\mathbf{x}, \mathbf{y} \in \mathbf{S}$, and clearly $\mathbf{x} \parallel \mathbf{y}$. According to Thm. 4 of Nieminen[4] there are elements $\mathbf{c}^*$ and $\mathbf{d}^*$ with $mub\ \{c, d\} = mub\ \{c^*, d^*\}$ and $\mathbf{c}^*, \mathbf{d}^* \prec \mathbf{x}, \mathbf{y}$, and hence we assume for simplicity that $\mathbf{c}, \mathbf{d} \prec \mathbf{x}, \mathbf{y}$ in this proof. Let us consider the element a. According to Lemma 2 above, $\mathbf{a} \leqslant \mathbf{k} \in mlb\ \{x, y\}$ or $\mathbf{a} \geqslant \mathbf{s} \in mub\ \{c, d\}$. If $\mathbf{a} \geqslant \mathbf{s} \in mub\ \{c, d\}$. then $\mathbf{b} \not\geqslant \mathbf{t} \in mub\ \{c, d\}$, because otherwise $\mathbf{a} \wedge \mathbf{b} > \mathbf{c}, \mathbf{d}$ or $\mathbf{a} \wedge \mathbf{b} = \mathbf{c}$ or $\mathbf{a} \wedge \mathbf{b} = \mathbf{d}$, and thus c and d cannot simultaneously belong to A; a contradiction. Now Lemma 2 and $\mathbf{b} \not\geqslant \mathbf{t} \in mub\ \{c, d\}$ imply that $\mathbf{b} \leqslant \mathbf{v} \in mlb\ \{x, y\}$, and so $\mathbf{b} < \mathbf{a}$, which contradicts the assumption $\mathbf{a} \parallel \mathbf{b}$. If $\mathbf{a} \leqslant \mathbf{k} \in mub\ \{c, d\}$, we obtain analogously two contradictions. Thus the possibility that S contains exactly one element is the only one, and hence A is a lattice as a poset.

Theorem 4—A finite modular 0, 1-poset $\mathbf{P}_{01}$ is algebraically semimodular.

PROOF : The property (2) of algebraic semimodularity follows from Theorem 5 of Nieminen[4] reported in the proof of Lemma 2. Thus it remains to prove the property (1). Let $\mathbf{a} \parallel \mathbf{b}$ and $\mathbf{a} \wedge \mathbf{b} < \mathbf{x} < \mathbf{a}$. According to Lemma 3 the set $\mathbf{A} = \{\mathbf{y} \mid \mathbf{y} \in \mathbf{P}_{01}$ and $\mathbf{a} \wedge \mathbf{b} \leqslant \mathbf{y} \leqslant \mathbf{a} \vee \mathbf{b}\}$ is a lattice, and as a subposet of a modular 0, 1-poset it is a modular lattice. As well known, the property (1) holds in every modular lattice, and thus the theorem follows.

Ore[5] has shown that in a finite semimodular poset every two maximal chains between two elements a and b, $\mathbf{a} < \mathbf{b}$, are of the same length (i. e. the Jordan-Hölder chain condition holds). By Theorem 1 every algebraically semimodular 0, 1-poset is also semimodular and satisfies thus the Jordan Hölder chain condition. The height $\mathbf{h}(\mathbf{a})$ of an element a in a finite $\mathbf{P}_{01}$ means the length of a longest maximal chain between 0 and a, and in a semimodular 0,1-poset $\mathbf{h}(\mathbf{a})$ is the length of any maximal chain between 0 and a. Now we can prove a connection between the algebraic semimodularity and the height.

Theorem 5—Let $\mathbf{P}_{01}$ be an algebraically semimodular 0,1-poset. The inequality $\mathbf{h}(\mathbf{a}) + \mathbf{h}(\mathbf{b}) \geqslant \mathbf{h}(\mathbf{a} \vee \mathbf{b}) + (\mathbf{a} \wedge \mathbf{b})$ holds in every χ-lattice derived from $\mathbf{P}_{01}$ for each pair $\mathbf{a}, \mathbf{b} \in \mathbf{P}_{01}$.

PROOF : Let $\mathbf{a}, \mathbf{b} \in \mathbf{P}_{01}$. If a and b are comparable, there is nothing to prove. Hence we assume that $\mathbf{a} \parallel \mathbf{b}$, and thus $\mathbf{a}, \mathbf{b} > \mathbf{a} \wedge \mathbf{b}$. Let $\mathbf{a} \wedge \mathbf{b} = \mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_r = \mathbf{a}$

and $a \wedge b = t_0, t_1, \ldots, t_s = b$ be two maximal chains. Because $a \wedge b \prec p_1, t_1$, there is by Theorem 1 an element $d_1 \succ p_1, t_1$. Similarly, Theorem 1 and $p_1 \prec p_2, d_1$ imply the existence of an element $d_2 \succ p_2, d_1$. By repeating this process we obtain finally an element $d_r \succ a, d_{r-1}$. Assume that no element d_r obtained from t_1 and any maximal chain from $a \wedge b$ to a is between a and $a \vee b$. Then $d_r, a \vee b \in mub\ \{a, t_1\}$. Because $a \prec d_r$, the property (2) implies that $a \prec a \vee b$, and then the lengths $h(a \vee b) - h(a)$ and $h(b) - h(a \wedge b)$ of the maximal chains from a to $a \vee b$ and from $a \wedge b$ to b, respectively, satisfy the relation $h(b) - h(a \wedge b) \geqslant h(a \vee b) - (a)$. Hence $h(a) + h(b) \geqslant h(a \vee b) + h(a \wedge b)$. If some d_r satisfies the relation $a \prec d_r < a \vee b$, the proof above can be repeated by starting with chains $t_1, d_1, \ldots, d_r$ and $t_1, t_2, \ldots, t_s = b$. As a final result we see that in any case $h(a \vee b) - h(a) \leqslant h(b) - h(a \vee b)$, which proves the assertion.

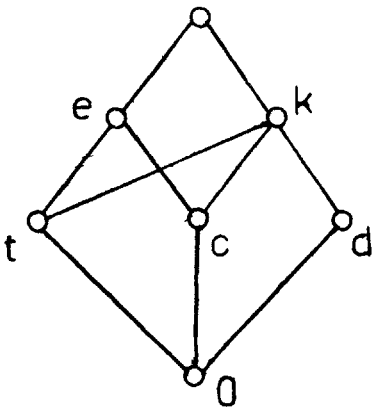

FIG. 1.

The 0, 1-poset P_{01} of Fig. 1 is algebraically semimodular as well as dually algebraically semimodular. Moreover, by a direct calculation one can see that the equation $h(a) + h(b) = h(a \wedge b) + h(a \vee b)$ holds in every χ-lattice derived from P_{01} for each pair $a, b \in P_{01}$. On the other hand, the poset P_{01} is nonmodular. This can be seen by putting $e \wedge k = t$ and by counting $e \wedge (d \vee (e \wedge c)) = e \wedge (d \vee c) = e \wedge k = c \parallel (e \wedge d) \vee (e \wedge c) = 0 \vee c = c$, which contradicts M_2. This example shows that the situation is more complicated in 0, 1-posets than in finite lattices L, where (1) and its dual are equivalent to the modularity as well as to the equation $h(a) + h(b) = h(a \wedge b) + h(a \vee b)$ for each pair $a, b \in L$. Unfortunately, we have not been able to prove or to disprove the converse of Theorem 5.

The author express his sincere thanks to the referee for his kind suggestions and comments.

REFERENCES

1. G. Birkhoff, *Lattice theory*, American Mathematical Society, Providence, Rhode Island 1973.
2. K. Leutola, and J. Nieminen, *Algebra Univ.* **16** (1983), 344–54.
3. J. Nieminen, *Yokohama Math. J.* **31** (1983), 13–20.
4. J. Nieminen *Indian J. pure appl. Math.* 16 (1985), 1425–31.
5. O. Ore, *Bull. Amer. Math. Soc.* **49** (1943), 558–66.