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A MODIFIED INSTABILITY CRITERION FOR HETEROGENEOUS SHEAR FLOWS

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Banerjee *et al.s*[1,2] unified instability criterion for heterogeneous shear flows is further modified by relaxing the very restrictive conditions on the curvature of the basic velocity profile in their work. As a consequence Howard's[3] semicircle region in which the complex wave velocity of an arbitrary unstable mode must lie is further reduced.

1. INTRODUCTION

In the stability problem of heterogeneous shear flows Banerjee *et al.*[1,2] established an independent necessary condition for instability which simultaneously gives Miles[4] criterion and a range of c_r and c_i for an unstable mode for a class of velocity and density distributions satisfying

(i) $a = U_{\min} > 0$ and

(ii) $(\rho U')' > 0$ or $(\rho U')' < 0$ everywhere in the flow domain, where ρ is the density, U is the basic velocity profile, $c = c_r + ic_i$ is the complex wave velocity and primes denote differentiation w.r.t, y. Similar results were derived by Gupta *et al.*[5] without the restriction (i) on the basic velocity profile. All these results have the inherent drawback that they cannot be applied to homogeneous shear flows for in this case condition (ii) in view of Rayleigh's[6] famous point of inflexion criterion implies that the system is stable. Therefore it becomes important to derive a necessary condition of instability which simultaneously gives Miles criterion, a range of c_r and c_i and also takes care of Rayleigh's point of inflexion criterion. The present paper is precisely in this direction. A modified instability criterion for heterogeneous shear flows in the foregoing sense is derived by relaxing the very restrictive conditions on the curvature of the basic velocity profile in the work of Banerjee *et al.*[1,2] As a consequence Howard's[3] semicircle region in which the complex wave velocity of an arbitrary unstable mode must lie is further reduced.

2. The Eigenvalue Problem and its Mathematical Analysis

The eigenvalue problem governing unstable modes in the stability problem of heterogeneous shear flows is given by[3]

$$[\rho (U - c) G']' - \left[\frac{(\rho U')'}{2} + k^2 \rho (U - c) + \frac{\rho \left(\frac{U'^2}{4} - g\beta \right)}{U - c} \right] G = 0 \qquad \ldots(1)$$

and

$$G = G(y) = 0 \text{ at } y = y_1 \text{ and } y = y_2 \qquad \ldots(2)$$

where $\rho = \rho(y)$ is the density, $U = U(y)$ is the basic velocity profile, $\beta = -\rho'/\rho > 0$, $\forall\ y \in [y_1, y_2]$, k is the wavenumber, $c = c_r + ic_i$ ($c_i > 0$) is the complex wave velocity and primes denote differentiation w. r. t. y.

We prove the following theorems :

Theorem 1—If (c, G) is a solution of eqns. (1) — (2) and

(i) $a = U_{\min} > 0$ and

(ii) $f(y) = \left[(\rho U')' + \frac{2\rho_{\min}\, b\pi^2}{(y_2 - y_1)^2} \right] > 0, \qquad \forall\ y \in [y_1, y_2],$

then

$$c_i^2 \leqslant \lambda \left(c_r - \frac{a}{m+1} \right), \qquad \ldots(3)$$

where

$$\lambda = \max_{[y_1, y_2]} \left\{ \frac{2\rho (m+1) \left(\frac{U'^2}{4} - g\beta \right)}{f(y)} \right\} \text{ and } m = b/a > 1 \text{ with}$$

$b = U_{\max}$,

Proof : Multiplying (1) by G^* (the complex and conjugate of G) and integrating over the range of y, we have upon integration by parts and making use of (2),

$$\int_{y_1}^{y_2} \rho (U - c) [\,|G'|^2 + k^2 |G|^2]\, dy + \tfrac{1}{2} \int_{y_1}^{y_2} (\rho U')' |G|^2\, dy + \int_{y_1}^{y_2} \frac{\rho \left(\frac{U'^2}{4} - g\beta \right)}{U - c} |G|^2\, dy = 0 \qquad \ldots(4)$$

Equating the real and imaginary parts of (4) to zero and cancelling c_i (> 0), we get

$$\int_{y_1}^{y_2} \rho (U - c_r) [\,|G'|^2 + k^2 |G|^2]\, dy + \tfrac{1}{2} \int_{y_1}^{y_2} (\rho U')' |G|^2\, dy + \int_{y_1}^{y_2} \frac{\rho (U - c_r) \left(\frac{U'^2}{4} - g\beta \right)}{(U - c_r)^2 + c_i^2} |G|^2\, dy = 0, \qquad \ldots(5)$$

$$\int_{y_1}^{y_2} \rho \left[\mid G' \mid^2 + k^2 \mid G \mid^2 \right] dy + \int_{y_1}^{y_2} \frac{\rho \left(g\beta - \frac{U'^2}{4} \right)}{(U - c_r)^2 + c_i^2} \mid G \mid^2 dy = 0. \quad \ldots (6)$$

Multiplying (6) by mc_r and adding the resulting equation to (5), we get

$$\int_{y_1}^{y_2} \rho \left[U + (m-1)\, c_r \right] \left(\mid G' \mid^2 + k^2 \mid G \mid^2 \right) dy + \int_{y_1}^{y_2} (\rho U')'/2 \mid G \mid^2 dy$$

$$+ \int_{y_1}^{y_2} \frac{\rho \left[U - (m+1)\, c_r \right] \left(\frac{U^2}{4} - g\beta \right)}{(U - c_r)^2 + c_i^2} \mid G \mid dy = 0 \qquad \ldots (7)$$

Since $a > 0$, and $a < c_r < b$ (Miles[4]), it follows that

$$[U + (m-1)\, c_r]_{\min} = b > 0 \text{ and } [U - (m+1)\, c_r]_{\max} = -a < 0. \quad \ldots (8)$$

Equation (7) upon using (8) and the Rayleigh-Ritz inequality[7]

$$\int_{y_1}^{y_2} \mid G' \mid^2 dy \geqslant \frac{\pi^2}{(y_2 - y_1)^2} \int_{y_1}^{y_2} \mid G \mid^2 dy \text{ (since } G(y_1) = 0 = G(y_2)), \qquad \ldots (9)$$

gives

$$\int_{y_1}^{y_2} \frac{f(y)\, c_i^2 + 2\rho \{U - (m+1)\, c_r\} \left(\frac{U'^2}{4} - g\beta \right)}{(U - c_r)^2 + c_i^2} \mid G \mid^2 dy \leqslant 0. \quad \ldots (10)$$

Under the conditions of the theorem, (10) clearly implies that

$$c_i^2 \leqslant \lambda \left(c_r - \frac{a}{m+1} \right)$$

where

$$\lambda = \max_{[y_1, y_2]} \left\{ \frac{2\rho (m+1) \left(\frac{U^2}{4} - g\beta \right)}{f(y)} \right\},$$

and this establishes the theorem.

Theorem 2—Under the conditions of Theorem 1, if

$$\lambda < \left(\frac{m-1}{m+1}\, a + b \right) - \left[\left(\frac{m-1}{m+1}\, a + b \right)^2 - (b-a)^2 \right]^{1/2} \qquad \ldots (11)$$

then the parabola

$$c_i^2 = \lambda \left(c_r - \frac{a}{m+1} \right), \qquad \ldots (12)$$

intersects Howard's semicircle

$$\left(c_r - \frac{a+b}{2} \right)^2 + c_i^2 = \left(\frac{b-a}{2} \right)^2 \qquad \ldots (13)$$

PROOF: It is easily seen that the parabola (12) touches Howard's semicircle (13), if

$$\lambda = \lambda_c = \left(\frac{m-1}{m+1} a + b \right) \pm \left[\left(\frac{m-1}{m+1} a + b \right)^2 - (b-a)^2 \right]^{1/2} \quad \ldots(14)$$

The value of λ_c given by (14) with positive sign is rejected as it leads to $c_r < a$ which violates $a < c_r < b$. Therefore

$$\lambda_c = \left(\frac{m-1}{m+1} a + b \right) - \left[\left(\frac{m-1}{m+1} a + b \right)^2 - (b-a)^2 \right]^{1/2}$$

Hence, if

$$\lambda < \left(\frac{m-1}{m+1} a - b \right) - \left[\left(\frac{m-1}{m+1} a + b \right)^2 - (b-a)^2 \right]^{1/2}$$

then the parabola (12) certainly intersects Howard's semicircle (13). This establishes the theorem.

Theorem 3—If (c, G) a solution of eqns. (1)—(2) and

(i) $a = U_{\min} > 0$

and

(ii) $g_1(y) = \left[(\rho U')' - \dfrac{2\rho_{\max} \pi^2 a}{(y_2 - y_1)^2} \right] < 0, \ \forall\ y \in [y_1, y_2]$

then

$$c_i^2 \leqslant \lambda^* \left(c_r + \frac{b}{m-1} \right) \quad \ldots(15)$$

where $\lambda^* = \max\limits_{[y_1, y_2]} \left\{ \dfrac{2\rho (m-1)(U'^2/4 - g\beta)}{|g_i(y)|} \right\}$ and $m = b/a > 1$

PROOF : Multiplying (6) by $-m c_r$, adding the resulting equation to (5) and proceeding as in Theorem 1, we get the result.

Theorem 4—Under the conditions of Theorem 3, if

$$\lambda^* < \left(\frac{m+1}{m-1} b + a \right) - \left[\left(\frac{m+1}{m-1} b + a \right)^2 - (b-a)^2 \right]^{1/2} \quad \ldots(16)$$

then the parabola

$$c_i^2 = \lambda^* \left(c_r + \frac{b}{m-1} \right) \quad \ldots(17)$$

intersects Howard's semicircle (13).

PROOF : Follows by proceeding as in Theorem 2.

3. Concluding Remarks

It follows from Theorem 1 or 3 that a necessary condition for instability is that $J = g(\beta)/U'^2 < \frac{1}{4}$ somewhere inside the flow domain (Miles[4]). Further Theorem 1—4 in conjunction with Howard's semicircle clearly show that if

$$a > 0 \text{ and either } f(y) > 0 \text{ or } g_1(y) < 0, \ \forall\ y \in [y_1, y_2] \qquad \ldots(18)$$

then under the conditions (11) or (16), Howard's semicircle bound for heterogeneous shear flows can further be reduced. The importance of condition (18) lies in the fact that under this condition the basic velocity profile can change the sign of its curvature in the flow domain and thus could derive according to Rayleigh's point of inflexion criterion possibly some instability characteristics which is quite relevant in the context in which the above theorems are proved. Further it could be noted that the above investigation is uniformly applicable to the instability problem of homogeneous shear flows also and the consequent reduction of Howard's semicircle bound for the complex wave velocity of an arbitrary unstable mode is first of its kind for the problem to the best of our knowledge and has been reported elsewhere.

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