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FIXED POINTS ON STAR-SHAPED SUBSETS OF CONVEX METRIC SPACES

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Let T be a Kannan mapping of a bounded closed star-shaped subset K of a convex metric space X, into itself and for every closed star-shaped subset F of K, $\sup_{y \in F} d(y, Ty) < \frac{1}{2}$ (diameter of F). It is proved that if there exists a minimal T-invariant star-shaped subset K^* of K then T has a unique fixed point in K.

Let X be a metric space. A mapping $W : X \times X \times [0, 1] \to X$ is said to be a convex structure on X, if for all x, y in X and $\lambda \in [0, 1]$, the following condition is satisfied

$$d\{u, W(x, y, \lambda)\} \leqslant \lambda\, d(u, x) + (1 - \lambda)\, d(u, y).$$

for all $u \in X$.

A metric space with convex structure is called convex metric space. A subset K of a convex metric space X is said to be convex if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$. The set K is said to be star-shaped if there exists $x_0 \in K$ such that $W(x, x_0, \lambda) \in K$ for all $x \in K$, $\lambda \in [0, 1]$. Clearly star-shaped subsets of X contain all convex subsets of X as a proper subclass. For examples and other details we refer to Takahashi[5], Guay *et al.*[2] and Halpern[3].

A mapping T of a convex metric space X into itself is said to be Kannan mapping on K if

$$d(Tx, Ty) \leqslant \frac{d(x, Tx) + d(y, Ty)}{2}$$

for all $x, y \in K$.

Dotson[1] has proved that a non-expansive mapping of a compact star-shaped subset K of a Banach space X into itself has a fixed point in K. Guay, *et al.*[2] succeeded to generalize it to the convex metric spaces. They used the fact that the mapping $T_{\lambda, x_0}(x) = \lambda T(x) + (1 - \lambda) x_0$ where $x, x_0 \in K$, $\lambda \in (0, 1)$, is a contraction if T is a non-expansive mapping. We prove a similar result for Kannan mappings. In this case, however, it is worthwhile to point out that T_{λ, x_0} may not be a Kannan mapping even if T is: Let T be the mapping on the unit interval defined by

$$T(x) = \begin{cases} 1 - x & x \in [0, \frac{1}{3}) \\ \dfrac{x+1}{3} & x \in (\frac{1}{3}, 1]. \end{cases}$$

Then T is a Kannan mapping where as $T_{\frac{1}{2}, 0}$ is not.

Theorem —Let T be a Kannan mapping of a non-empty bounded closed star-shaped subset K of a convex metric space X, into itself. Suppose $\sup_{y \in F} d(y, Ty) < \frac{1}{2}\delta(F)$. $\{\delta(F)$ being diameter of $F\}$ for every closed T-invariant star-shaped subset F of K with non-zero diameter. Then T has a unique fixed point in K if there exists a minimal closed T-invariant star-shaped subset K^* of K.

PROOF : If $\delta(K^*) = 0$ then the point in K^* is fixed point. Suppose $\delta(K^*) > 0$. It follows that there exists $x_0 \in K^*$ such that for each $x \in K^*$, $\lambda \in [0, 1]$, $W(x, x_0, \lambda) \in K^*$. For any $y \in K^*$, we have

$$d(Ty, Tx_0) \leqslant \frac{d(y, Ty)}{2} + \frac{d(x_0, Tx_0)}{2} \leqslant \sup_{y \in K^*} d(y, Ty).$$

Hence $T(K^*)$ is contained in a closed sphere C with Tx_0 as centre and $\sup_{y \in K^*} d(y, Ty)$ as radius, $K^* \cap C$ is T-invariant star-shaped. By minimality of K^* it follows that $K^* \subset C$. Therefore

$$\sup_{y \in K^*} d(y, Tx_0) \leqslant \sup_{y \in K^*} d(y, Ty).$$

Let

$$K' = \{Z \in K^* : \tfrac{1}{2} \sup_{y \in K^*} d(Z, y) \leqslant \sup_{y \in K^*} d(y, Ty)\}.$$

For $y \in K^*$

$$d(x_0, y) \leqslant d(x_0, Tx_0) + d(Tx_0, y) \leqslant 2 \sup_{y \in K^*} d(y, Ty).$$

This implies that $x_0 \in K'$.

For $z \in K'$

$$d(Tz, y) \leqslant d(Tz, Tx_0) + d(Tx_0, y)$$

$$\leqslant \sup_{y \in K^*} d(y, Ty) + \sup_{y \in K^*} d(y, Ty).$$

This implies that K' is T-invariant.

For $x \in K'$, $y \in K^*$, and $\lambda \in [0, 1]$. we have

$$d\{W(x, x_0, \lambda), y\} \leqslant \lambda\, d(x, y) + (1 - \lambda)\, d(x_0, y)$$

$$\leqslant 2 \sup_{y \in K^*} d(y, Ty),$$

that is

$$\tfrac{1}{2} \sup_{y \in K^*} d\{W(x, x_0, \lambda), y\} \leqslant \sup_{y \in K^*} d(y, Ty).$$

It follows that $W(x, x_0, \lambda) \in K'$ for all $x \in K'$ and $\lambda \in [0, 1]$. Hence K' is star-shaped.

Next, suppose that z belongs to closure of K'. Then there exists a sequence (z_n) in K' such that $z_n \to z$, and

$$\tfrac{1}{2} \sup_{y \in K^*} d(z_n, y) \leqslant \sup_{y \in K^*} d(y, Ty).$$

Letting n tend to infinity, we have

$$\tfrac{1}{2} \sup_{y \in K^*} d(z, y) \leqslant \sup_{y \in K^*} d(y, Ty).$$

It follows that $z \in K'$. From our hypothesis we have

$$\delta(K') \leqslant 2 \sup_{y \in K^*} d(y, Ty) < \delta(K^*).$$

Hence K' is a proper closed T-invariant star-shaped subset of K^*, which contradicts the minimality of K^*. The uniqueness of the fixed point is easy to verify.

This completes the proof of the theorem.

Remark : The above theorem also enables us to relax the condition of reflexiveness on X used by Kannan[4]. This is quite significant as, in practice, we encounter many important spaces, that are not reflexive.

In the following, we give an example of a non-reflexive space admitting a fixed point of Kannan mapping.

Example —Consider the non-reflexive Banach space c of all convergent sequences (ξ_i) of real numbers with norm defined by $\|(\xi_i)\| = \sup_i |\xi_i|$. Then the set $K = \{(\xi_i) \in c :$

$\tfrac{1}{3} \leqslant \xi_i \leqslant 1$ and $(\xi_i - \tfrac{1}{2})^2 \leqslant (\xi_{i+1} - \tfrac{1}{2})^2\}$ is nonempty bounded closed star-shaped subset of X. The Kannan mapping $T: K \to K$ is defined by $T(\xi_i) = \dfrac{\xi_i + 1}{3}$. The set $K^* = \{(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \ldots)\}$ is a minimal closed T-invariant star-shaped subset of K. Then obviously $(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \ldots)$ is the unique fixed point of T.

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