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SANDWICH PLATE VIBRATIONS—AN INTERACTING CONTINUUM APPROACH

(IN MEMORIUM PROFESSOR KALI VISWANATHAM)

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A symbolic equation governing sandwich plate vibrations has been derived adapting the interacting continuum approach of Bache and Hegemier[16,20]. It is further shown that the exact frequency equation for harmonic waves in sandwich plates obtained by Lee and Chang[7] can be recovered as a special case of the present general equation. The present theory is also compared with the exact theory in the case of symmetric SH waves.

1. INTRODUCTION

Sandwich plates are widely used structural elements in a variety of industries. A large number of studies appear in literature concerning the mechanical behaviour of these very useful components of modern structures. This paper is an attempt at appreciating the dynamical behaviour of symmetric sandwich plates from a particular stand point of view.

Yu's[1] work on the vibrations of sandwich plates marks a beginning of a systematic study. Earlier to Yu the studies were concerned with static problems and further they were limited by the assumption that the faces of the plate were thinner, stiffer and heavier than the core. In an investigation carried out by Reissner[2] the faces were taken to be membranes, although the transverse-shear deformation in the core was taken care of. Hoff[3] later considered also the flexural rigidity of the faces, and Eringen[4] further included in his investigation the flexural rigidity of the core, but the transverse-shear deformation in the faces has never been taken into consideration in the sandwich plate problem by any author previous to Yu. Then, the investigation had been limited to the bending and buckling of such plates.

Yu[1] presented an exploratory study aiming at attacking the vibration problem of sandwich plates for the one-dimensional plane-strain case. Guided by a knowledge on the vibration of homogeneous plates (Mindlin[5]) he considered the effects of transverse-shear deformation and rotatory inertia in both the core and the faces. He removed the restriction on the magnitude of the ratios between the thicknesses, material densities, and elastic constants of the core and faces. The treatment is therefore more inclusive than any other study of sandwich plate vibrations. A new set of equilibrium equations is obtained as a by-product, which is more complete than, and reducible to,

previous bending equations such as those given by Reissner, Hoff and Eringen. The method of approach used was an extension of the one employed by Mindlin[5] in a systematic treatment of the vibration of homogeneous plates, and the results for the sandwich plates reduce to those of his for the homogeneous plates as special limiting cases. To illustrate the use of the results of the theory, he solved two problems, namely, the problem of bending of a cantilever plate subjected to loading at the unsupported end, and the problem of propagation of straight-creasted waves in an infinite plate. Later Yu[6] extended this study to free flexural vibrations of sandwich plates and to simply supported, finite sandwich plates. In recent times Lee and Chang[7] have analysed harmonic wave motion in symmetric sandwich plates of arbitrary thicknesses of core and facing, in the frame work of the exact 3 dimensional elasticity theory.

A sandwich plate is essentially a layered composite and wave motion in layered composites has been extensively studied by several workers from several different points of view. Two classes of the layered composites are usually considered in these studies. They are (1) media made of an arbitrary number of alternating plane parallel layers of two homogeneous elastic materials; and (2) media consisting of multilayered shells and plates. In the case of the former class many approximate theories such as the effective modulus theory (Pastma[8], Rytov[9]), the effective stiffness theory (sun, Achenbach *et al.*[10], Drumheller and Bedford[11]), cantinuum mixture theories (Stern and Bedford[12]), Floquet wave theory in conjunction with elasticity solutions (Delpha, and Herrmann[13]. Datta, Shah *et al.*[14]), interacting continunm theories coupled with asymptotic expansions (Hegemier and Nayfeh[15], Hegemier and Bache[16]), and the reccnt effective dispersion theory (Delpha and Herrmann[13])—have been constructed to describe their dynamical behaviour.

In the case of the other class recently Green and Naghdi[17] (see their paper for a short survey of work in this area) developed a linear theory of laminated composite plates, containing N-laminates each consisting of a plate of homogeneous orthotropic material, extending their earlier theory of plates for single phase media.

Theoretically any of the above dynamical theories can be employed to study the vibrations of sandwich plates, for instance similar to the one considered by Lee and Chang[7]. In this connection it is interesting to note the remarks of Durocher and Solecki[18]. "Three-layered plates are generally classified as being either of the sandwich or laminated type, where a sandwich plate usually consists of a relatively thick, low-density, low-stiffness core bonded between two thin sheets of high strength material and a laminated plate in normally constructed using several layers whose geometric and material characteristics are some what similar. Since their configurations are quite different, laminated and sandwich plates are usually analysed on the basis of widely differing assumptions". This paper is an application of techniques developed in laminated media to sandwich plate vibrations. Recently Green and Naghdi[17] derived in the limit of vanishingw [see eqn. (7)], the results of Lee and Chang[7] for the isotropic case and Dravinski[19] has studied the propagation of elastic waves in a plate of finite thickness

and infinite length with periodic structure of m different layers. Dravenski's[19] analysis was, however, restricted to antiplane strain. We show that the exact frequency equation, a vanishing conditions of an 18 × 18 determinant for three dimensional harmonic waves in a sandwich plate, obtained by Lee and Chang[7] can be recovered as a particular case of the symbolic equation derived in this paper by an extension of (Bache and Hegemier's[20] higher order theory for single plate vibrations in which they (Bache and Hegemier) employed an asymptotic expansion procedure similar to the one they adopted in their interacting continuum approach for laminated media (Hegemier and Bache)[16]. The motivation for employing, in particular, the results of Bache and Hegemier comes from the remark of Delph and Herrmann[13] that "the theory of Hegemier and co-workers takes on somewhat different approach from those cited previously and in fact, exactly reproduces the dispersion spectrum if all the terms in the infinite series are retained", as they compared their recent effective dispersion theory with the various existing ones.

In section 2 Bache and Hegemier's[20] higher order theory is extended to sandwich plate vibrations and in section 3, the extended theory is compared with the exact theory of Lee and Chang[7] in the case of symmetric SH wave propagation.

2. Problem Formulation

We consider an elastic sandwich plate which consists of three layers, namely a middle layer and two cover layers, each layer being homogeneous, isotropic and linearly elastic. The plate is of symmetric construction i. e., the facings are of the same material and the same thickness. We adopt the notation of Bache and Hegemier[20] with obvious minor modifications. The adjoining Figure 1 explains a good deal of the notation.

The problem is now to solve the equations of motion for free vibrations of the plate with no body forces together with interface and boundary conditions. We denote the displacement vectory by $\bar{u}^{(\alpha,k)}$ which can be split into its irrotational part $\bar{U}^{(\alpha,k)}$ and solonoidal part $\bar{V}^{(\alpha,k)}$ that satisfy, as a consequence of Navier's equations of motion.

$$\left(\frac{\partial^2}{\partial t^2} - C_L^{(\alpha)^2} \nabla^2\right) \bar{U}^{(\alpha,k)} = 0;\ \bar{\nabla} \times \bar{U}^{(\alpha,k)} = 0$$

$$\left(\frac{\partial^2}{\partial t^2} - C_T^{(\alpha)^2} \nabla^2\right) \bar{V}^{(\alpha,k)} = 0;\ \bar{\nabla}\ \bar{V}^{(\alpha,k)} = 0 \qquad \ldots(1)$$

where

$$\bar{\nabla} = \bar{e}_1 \frac{\partial}{\partial x_1} + \bar{e}_2 \frac{\partial}{\partial x_2} + \bar{e}_3^{(\alpha,k)} \frac{\partial}{\partial x_3^{(\alpha,k)}};\ \bar{e}_1, \bar{e}_2 \text{ and } \bar{e}_3^{(\alpha,k)}$$

are the unit vectors along x_1, x_2 and $x_3^{(\alpha,k)}$ axes,

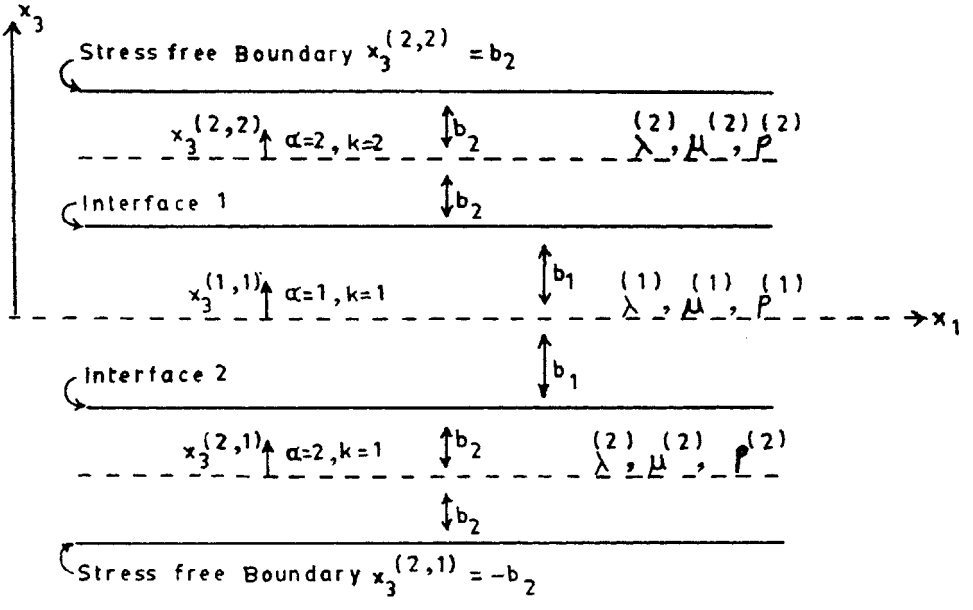

FIG. 1. A symmetric sandwich plate.

Core : $-b_1 \leqslant x_3 \leqslant b_1$ of material 1.

upperfacing : $b_1 \leqslant x_3 \leqslant b_1 + 2b_2$ of material 2,

lower facing : $-(b_1 + 2b_2) \leqslant x_3 \leqslant -b_1$ of material 2.

x_3 is thickness co-ordinate normal to the plate.

$x_3 = 0$ is midplane of core

$x_3^{(\alpha,k)}$ is local thickness co-ordinate.

$x_3^{(\alpha,k)} = 0$ is midplane of (α, k) laminate.

(α,k) | (2,2) upper facing
(1,1) core
(2,1) lower facing

$$\nabla^2 = \bar{\nabla} \cdot \bar{\nabla}, \; C_L^{(\alpha)^2} = \frac{\lambda^{(\alpha)} + 2\mu^{(\alpha)}}{\rho^{(\alpha)}}, \; C_T^{(\alpha)^2} = \frac{\mu^{(\alpha)}}{\rho^{(\alpha)}}$$

Expanding $\bar{U}^{(\alpha,k)}(x_1, x_2, x_3^{(\alpha,k)}, t)$ and $\bar{V}^{(\alpha,k)}(x_1, x_2, x_3^{(\alpha,k)}, t)$

substituting in (1) and incorporating the recurrence relations obtained by equating like powers of $x_3^{(\alpha,k)}$ together with the implications of the solonoidal and irrotational conditions in eqn (1) we get expressions for the displacements, while the stresses can be calculated via Hookes' law.

As we consider the problem of free vibrations we have the following conditions. The boundaries are stress free, i. e.

$$\sigma_{3i}^{(2,1)} = 0 \quad \text{on } x_3^{(2,1)} = -b_2$$

$$\sigma_{3i}^{(2,2)} = 0 \quad \text{on } x_3^{(2,2)} = b_2.$$

The displacements and the stresses are continuous on the two interfaces, i. e.

$$u_1^{(2,1)}(x_3^{(2,1)} = b_2) = u_1^{(1,1)}(x_3^{(1,1)} = -b_1)$$

$$u_1^{(2,2)}(x_3^{(2,2)} = -b_2) = u_1^{(1,1)} \quad x_3^{(1,1)} = b_1)$$

$$u_2^{(2,1)}(x_3^{(2,1)} = b_2) = u_2^{(1,1)}(x_3^{(1,1)} = -b_1)$$

$$u_2^{(2,2)}(x_3^{(2,2)} = -b_2) = u_2^{(1,1)}(x_3^{(1,1)} = b_1)$$

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$$\sigma_{3i}^{(2,2)}(x_3^{(2,2)} = -b_2) = \sigma_{3i}^{(1,1)}(x_3^{(1,1)} = b_1)$$

$$i = 1, 2, 3. \qquad \ldots(2)$$

When the expressions for displacements and stresses are substituted in (2) we obtain eighteen equations in eighteen functions $U_{1(0)}^{(\alpha,k)}$ etc.

These eighteen equations can be written as a single matrix equation

$$Za = 0. \qquad \ldots(3)$$

Where Z is the 18×18 matrix and a is the 18×1 vector

$$\left(U_{1(0)}^{(1,1)} \quad \ldots \quad V_{2(1)}^{(2,2)} \right)^T.$$

The elements of Z involve infinite series in the operators $L_u^{(\alpha)}$ and $L_v^{(\alpha)}$ and (3) is the exact elasticity solution to the problem of free vibrations of the sandwich plate. A necessary and sufficient condition for the existence of a nontrivial a for (3) gives the symbolic frequency equation

$$\text{Det } z = 0. \qquad \ldots(4)$$

Equation (4) is the general symbolic equation governing the free vibrations of a sandwich plate. Since eqn. (3) involves partial differential equations of infinite order, approximations should be made for mathematical tractability of (4) on the lines of Bache and Hegemier leading to higer order theories of any desired order depending on the number of retained terms of the expansion.

3. Example : Symmetric SH Waves

For harmonic wave motion along x_1-direction the equation (4) coincides-when all the terms of the series are included with the one obtained by Lee and Chang. It can be easily demonstrated that the exact frequency spectra obtained by Lee and Chang in the individual cases of symmetric extensional motion in plane strain, antisymmetric flexural motion in plane strain, and symmetric and anti-symmetric *SH*-wave propagations are particular cases of (4) when all the terms of the infinite series are included in these individual cases. For instance in the sandwich plate we consider the relatively simple case of symmetric *SH*-wave propagation in plane strain in which we have

$$\begin{aligned} u_1(x_1, x_2, x_3, t) &= 0 \\ u_2(x_1, x_2, x_3, t) &= u_2(x_1, x_2 - x_3, t) \\ u_3(x_1, x_2, x_3, t) &= 0 \end{aligned} \qquad \ldots(5)$$

and for harmonic waves we write,

$$\begin{Bmatrix} V_{2(0)}^{(1,1)} \\ V_{2(0)}^{(2,2)} \\ V_{2(1)}^{(2,2)} \end{Bmatrix} = \begin{Bmatrix} A_1 \\ A_2 \\ -A_3 \end{Bmatrix} e^{i(\xi x_1 + \omega t)} \qquad \ldots(6)$$

and all the other constants being zeros. Substituting (6) in (4) leads to the frequency equation

$$2\mu_2\beta_2 \cos(\beta_1 b_1) \sin(\beta_2 b_2) \cos(\beta_2 b_2) + \mu_1 \beta_1 \sin(\beta_1 b_1) \cos^2(\beta_2 b_2) - \sin^2(\beta_2 b_2),$$

where

$$\beta_i^2 = \frac{\omega^2}{V_{2i}^2} - \xi^2 \ (i = 1,2). \qquad \ldots(7)$$

Equation (7) reduces – when b_2 in (7) is replaced by $b_2/2$ to

$$\mu_2 \beta_2 \cos \beta_1 b_1 \sin \beta_2 b_2 + \mu_1 \beta_1 \cos \beta_2 b_2 \sin \beta_1 b_1 = 0 \qquad \ldots(8)$$

which is the same as eqn. (13), Lee and Chang[7].

Equation (8) can be written, for symmetric *SH*-waves in plates of the following specifications (designated SP7)

$$\frac{E_2}{E_1} = 20.0,\ \gamma_1 = 0.3,\ \gamma_2 = 0.3,\ \frac{\rho_2}{\rho_1}\ 2.0,\ \frac{b_2}{b_1} = 0.1 \text{ as}$$

$$\tan [\pi/2\ (\Omega^2 - \zeta^2)^{1/2}] + 6.32 \tan [\pi/2 \times 0.1\ (0.1\ \Omega^2 - \zeta^2)^{1/2}] = 0 \quad ...(9)$$

where

$$\Omega = \frac{2\omega b_1}{\pi V_{21}} \text{ and } \zeta = \frac{2b_1}{\pi}.$$

For the first order theory, where the first two terms in the Tan functions' expansions are retained, we introduce compensation factors K^2 and L^2 for neglecting the higher order terms and write equation (9) as

$$(\Omega^2 - \zeta^2)^{1/2}\left[1 + \frac{K^2\pi^2}{12}(\Omega^2 - \zeta^2)\right] + 0.632\,[0.1\ \Omega^2 - \gamma^2)^{1/2}\left[1 + \frac{L^2\pi^2}{12} \times 0.01\ (0.1\ \Omega^2 - \gamma^2)\right] = 0. \quad ...(10)$$

These compensation factors are determined by matching the roots at $\zeta = 0$ and the slopes at $\zeta = 0$ (for $\zeta \ll 1$) for the two equations (9) and (10). The resulting compensation factors are noted as

$$K^2 = 0.1405229 \text{ and } L^2 = -3263.8077.$$

Figure 2 illustrates the frequency/wave number relation for the exact and the present first order theory.

4. Conclusions

A sandwich plate is considered as a three-layered composite and a higher order theory is proposed for the vibrations of a symmetric sandwich plate made of two isotropic materials. The interacting continuum approach of Bache and Hegemier is employed to derive the governing equation. This general symbolic equation is a 18 × 18 determental condition whose terms contain infinite series of the operators involved. When b_2 the thickness of the facings is taken to be zero, it can be shown that the present frequency equation reduces to that for a single plate (Achenbach[21]) in case of harmonic wave propagation and further when all the terms of the series are included theoretically, the equation derived here turns out to be the exact frequency equation. This fact is demonstrated by (eqn. 8), in the case of symmetric harmonic SH wave motion in the plate (Lee and Chang). The infinite series, for mathematical tractability, can be truncated as in the case of a single plate (Bache and Hegemier[20]) thus giving rise to theories

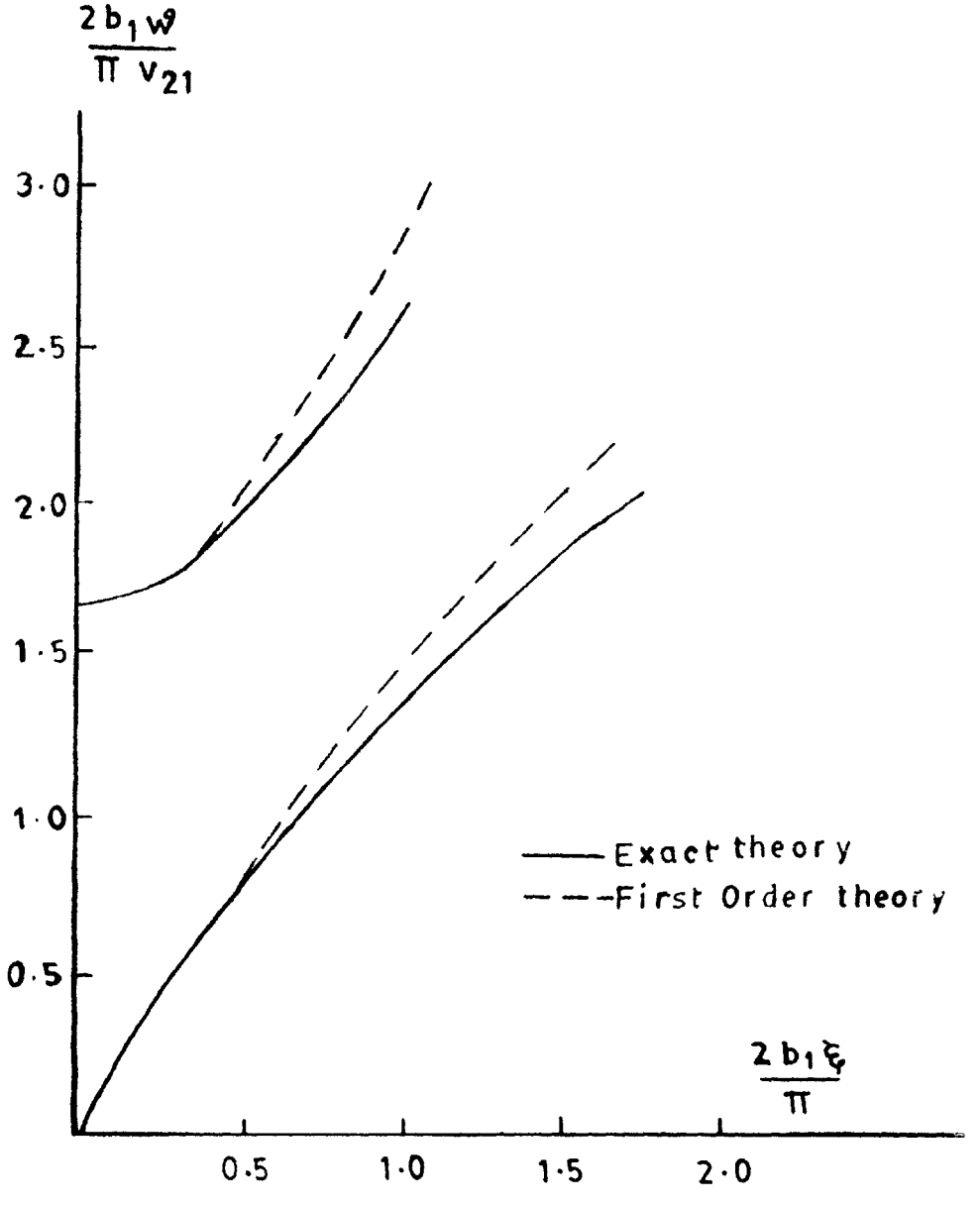

FIG. 2. Dispersion curves of symmetric SH waves in the sandwich plate SP 7.

of any desired higher order. An example of the theory is presented in the case of SH wave motion in plates designated SP7. The first order theory is compared with the exact theory in this case and the results are numerically illustrated. The analysis of the section 3, can be extended to all the other modes of vibration in sandwhich plates.

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