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ITERATIVE METHODS OF SOLUTIONS FOR LINEAR AND QUASI LINEAR COMPLEMENTARITY PROBLEMS

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The present work has been conceived out of a need to extend the method of Ahn to give an iterative procedure for approximating solution of a 'quasi-linear complementarity problem' (QLCP). We also give the bound for spectral radius of them odified matrix in the context of QLCP. Further we give more results on the modification of the algorithm of Pang for finding the solution of QLCP for a pair (M, q), where M is symmetric and positive definite. The fixed parameter approach of Pang also has been modified to incorporate the variable parameter method in successive iteration process.

1. INTRODUCTION

The numerical method for LCP carries with it two methods, e. g., direct and indirect methods. Because of the complexity, the use of direct method is restricted for large size problems. Therefore iterative methods are well suited for such problems. The present work attempts to develop the procedure for finding approximate solutions of Quasi-Linear-Complementarity problems by iterative technique. Essentially this extends the earlier algorithm of Ahn[1] used for solution method for Linear Complementarity problems. Secondly, we also give an extension of a method of Pang[5] to incorporate QLCP for the same purpose. Our attempt next would be to briefly indicate the essential procedure of Pang[5] because we would refer that latter in our extensions.

Consider the symmetric LCP (q, M) :

$$q + Mx \geqslant 0, \quad x \geqslant 0, \text{ and } x^T (q + Mx) = 0$$

where $q \in R^n$ and $M \in R^{n \times n}$ are given and $x \in R^n$. Let (B, C) be a Q-splitting of the matrix M, i. e. $M = B + C$ is a Q-matrix [the LCP (q, B) has a solution for all vectors q]. Let E^k be a non-negative diagonal matrix with $E^k_{ii} < 1$. Define the point to set algorithmic map A^k as follows: for all vectors x,

$$A^k (x^k) = \text{solution set of the LCP } (q + Cx, B, E^k x^k). \qquad ...(1.1)$$

The latter LCP (r, B, s) is to find y so that

$$r + By \geqslant 0, \; y \geqslant s \text{ and } (y - s)^T (r + By) = 0.$$

The LCP (r, B, s) can be converted into the LCP $(r + Bs, B)$ if we translate the variable $x = y - s$ since B is a Q-matrix, the set $A^k\ (x^k)$ is non-empty for all vectors x. Moreover a vector x^* solves the LCP (q, M) if and only if it is fixed point of the map A^k i. e. $x^* \in A^k\ (x^*)$.

We define an iterative technique for solving the LCP (q, M) given the diagonal matrix E^k and the Q-splitting (B, C) of the matrix M. Let $x^0 \geqslant 0$ be an arbitrary non-negative vector. In general $x^k \geqslant 0$, $k \geqslant 0$ let x^{k+1} be any vector in the set $A^k(x^k)$.

The motivation for using the map $A^k(x^k)$ lies in the fact that the matrix E^k may satisfy the bound $E^k_{ii} < 1$ after the iteration proceeds onwards after a fixed index k_0. This idea is compatible with the usual approach in the contraction mapping case where a fixed power of a mapping may be a contraction although the original map may not be a contraction.

If B is a P-matrix (A real matrix $A \in R^{n \times n}$ is said to be a P-matrix if it has positive principal minors) then the set $A^k(x^k)$ is singleton for all x. In this case, each x^{k+1} will be uniquely defined.

Pang[4] has given necessary and sufficient conditions on the matrix M on the convergence property, i. e. for all vector q and all starting vector $x^\circ \geqslant 0$, each sequence $\{x^k\}$ generated by the iterative technique will converge to some solution of the LCP (q, M).

A Quasi-linear Complementarity (QLCP) can be stated as follows

Find $z \in R^n$ such that

$$z - Qz \geqslant 0,\ Mz + q \geqslant 0\ (z - Qz)^T\ (Mz + q) = 0. \qquad \text{...(1.2)}$$

With splitting as indicated previously the point to-set algorithmic map takes the shape.

$$A^k\ (x^k) = \text{Solution set of LCP } (q + BQ\ x^k + C\ x^k,\ B,\ E^k\ (1 - Q)x^k) \qquad \text{...(1.3)}$$

Variantly x^* solves the QLCP (1.2) if and only if $(1 - Q)\ x^*$ is a point in range of the set valued map A, i. e. a point in $A\ (x^*)$. The result connected with the convergence of various dual iterative techniques for the solution of strictly convex quadratic program

$$\min_{(1-Q)x \geqslant 0} \quad f(x) = q^T\ x + \tfrac{1}{2}\ x^T\ Mx \qquad \text{...(1.4)}$$

can be derived by the methods of Pang[5].

We explain some matrix notations as follows : If A is an $n \times m$ matrix, α and β are subsets of $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$, respectively, by $A_{\alpha\beta}$ we denote the sub-matrix of A whose rows and columns are indexed by α and β respectively. If $\alpha = \{1, \ldots, n\}$, we denote by A_β the submatrix whose columns of A are indexed by β; similar definition applies to A_α.

2. PRELIMINARIES

We restate again the QLCP as :

find $z \in R^n$, such that

$$z - Qz \geqslant 0,\ Mz + q > 0,\ (z - Qz)^T (Mz + q) = 0 \qquad \text{...(2.1)}$$

where M is an $n \times n$ real and non-symmetric matrix, q is $n \times 1$ vector. If we take $Q = 0$ in QLCP we get the same LCP as in Ahn[1].

First of all we describe the notations which occur in the QLCP. All matrices and vectors are real. A matrix A with m-rows, n-columns is denoted by $R^{m \times n}$. Row i of matrix A is denoted by A_i and column j by A_j and the element in row i and column j by A_{ij}. The transpose of a matrix is denoted by super script T, such as the transpose of the matrix A is given by A^T, $|A|$ denotes the matrix obtained from the real matrix $A \in R^{m \times n}$ by replacing each element A_{ij} by its absolute value.

If $x \in R^n$, x_+ denotes the vector with elements

$$(x_+)_j = \max\{0, x_j\};\ j = 1, 2, \ldots, n.$$

For any x and y in R^n, it can be easily shown that

(i) $(x + y)_+ \leqslant x_+ + y_+$,

(ii) $x \leqslant y \Rightarrow x_+ \leqslant y_+$.

A real matrix $A \in R^{n \times n}$ is said to be a z-matrix (a P-matrix) if it has non-positive off diagonal entries (positive primal minors).

A square matrix with non-positive off diagonal elements and wiih a non-negative inverse in called an M-matrix. It can be easily shown that a matrix which is both a Z-matrix and a P-matrix is an M-matrix (or Minkowski matrix).

Given any real matrix $A \in R^{n \times n}$, we define its comparison matrix

$$A_c = (C_{ij})$$

by

$$C_{ii} = |A_{ii}|$$

and

$$C_{ij} = -|A_{ij}|,\ i \neq j,\ i, j = 1, 2, \ldots, n$$

This definition is due to Verga[6]

3. ITERATIVE ALGORITHM

For solving QLCP (2.1) we describe the general fundamental algorithm.

Lemma 3.1—Let $M \in R^{n \times n}$ and E be any positive diagonal matrix, then,

$$z - Qz \geqslant 0,\ Mz + q > 0,\ (z - Q\,z)^T\,(Mz + q) = 0$$

$$\Leftrightarrow z = \{(1 - Q)\,z - \omega\,E\,(M\,z + q)\}_+, \text{ for all or some } \omega > 0.$$

Its proof is same as in Mangasarian[2]. This result can be transformed to a fixed point problem for solving the equation $z = f(z)$

where

$$f(z) = \{(1 - Q)z - \omega E\,(Mz + q)\}_+.$$

This result readily leads to the following general algorithm suggested by Mangasarian[2]. We modify this algorithm at certain steps.

Algorithm 3.1—Let $z^0 > 0$, compute

$$z^{k+1} = \lambda\,[(1 - Q)\,z^k - \omega E^k\,(Mz^k + q + K^k\,(1 - Q)\,(z^{k+1} - z^k))]_+ + (1 - \lambda)\,(1 - Q)\,z^k \quad \text{...(3.1)}$$

where

$$k = 0, 1, \ldots\ 0 < \lambda \leqslant 1,\ \omega > 0$$

and $\{E^k\}$ and $\{K^k\}$ are bounded sequences of matrices in $R^{n \times n}$, with each E^k being a positive diagonal matrix satisfying $E^k > \alpha\,I$, for some $\alpha > 0$

where I is the identity matrix.

For the symmetric case Mangasarian has established convergence criteria of this general algorithm. We simplify this algorithm by setting

$$\lambda = 1,\ E^k = E,\ K^k = K, \text{ for each } k.$$

Remark : As has been indicated in the conclusion, we can relax the above criteria for fixing the matrix powers E^k and K^k as constant matrices to derive certain variable parameter algorithm as well.

Algorithm 3.2—Let $z^0 > 0$, compute,

$$z^{k+1} = [(1 - Q)\,z^k - \omega E\,(Mz^k + q + K\,(1 - Q)\,(z^{k+1} - z^k))]_+$$

$$k = 0, 1, \ldots \quad \text{...(3.2)}$$

where $\omega > 0$, E is a positive diagonal matrix and K is either strictly upper triangular or lower triangular matrix. Convergence properties for non-symmetric situations can not be established relying on the descent function of the form

$$\tfrac{1}{2}\,x^T\,M\,x + q^T x$$

so the recurssive relation between two successive iterations will be utilized here.

4. Convergence Properties

First of all we develop the fundamental recursive inequality for Algorithm 3.2 which will be the basis of convergence. This inequality is derived from the inequality properties of x_+ and y_+.

Lemma 4.1—The kth and $(k+1)$ th solutions z^k and z^{k+1} satisfy the partial ordering recurssive inequality :

$$| z^{k+1} - z^k | \leqslant (I - \omega E | K||1 - Q |)^{-1} | (1 - Q)(I + \omega EK) - \omega EM||z^k - z^{k-1} | . \quad ...(4.1)$$

From this Lemma we can produce a condition for the sequence $\{z^k\}$ of Algorithm 3.2 to be bounded and have an accumulation point which solves the QLCP (2.1).

If we put $Q = 0$ in (4.1) we have

$$| z^{k+1} - z^k | \leqslant (I - \omega E | K |)^{-1} | I - \omega E (M - K)||z^k - z^{k-1} |$$

which is the standard form given by Ahn[1].

Theorem 4.1—Suppose that the given iteration parameter ω, E, K and the underlying matrix M satisfy

$$\mu ((I - \omega E | K||1 - Q |)^{-1} | (1 - Q) (I + \omega EK) - \omega EM |) < 1 \quad ...(4.2)$$

where μ (.) denotes the spectral radius; then the sequence $\{z^k\}$ of Algorithm 3.2 converges to a solution z^* of QLCP.

The proof is similar to Ahn[1].

Here also we can find the same spectral radius which is established by Ahn[1], simply by taking $Q = 0$, in (4.2), viz.,

$$\mu ((I - \omega E | K |)^{-1} | I - \omega E (M - K) |) < 1.$$

We shall start the next section in which we modify the same result on the convergence of iterative methods for the symmetric QLCP. Pang had developed necessary and sufficient condition (for a fixed parameter) for the convergence of iterative method and for solving each individual LCP. We shall extend the method of Pang to incorporate it for the treatment of QLCP.

5. Non-degenerate Case

We classify our analysis into two cases which depends on the nature of the matrix, i. e. either the matrix is non-degenerate or positive semi-definite. Since we know that the matrix M is non-degenerate if all its principal minors are non-zero and the same case for the non-degeneracy in linear complementarity theory[3] the matrix M is non-degenerate if and only if the LCP (q, M) has a finite number of solutions for all vectors q.

The following theorem is the main result of this section.

Theorem 5.1—Let M be symmetric and non-degenerate matrix. Let (B, C) be regular Q-splitting of the matrix M. Let E^k be a non-negative diagonal matrix, with $E^k_{ii} < 1$, for all i. Then the following statements are equivalent:

(A) for some vector q and any initial vector $x^0 \geqslant 0$, any sequence $\{x^k\}$ satisfying $(1 - Q)\, x^{k+1} \in A^k\,(x^k)$ is bounded and thus has at least one accumulation point, moreover, any such point solves the QLCP (q, M).

(B) for some vector q, the quadratic function $f(x) = q^T x + \frac{1}{2} x^T Mx$ is bounded below for $(1 - Q)\, x \geqslant 0$.

(C) for some vector q and any initial vector $x^0 > 0$, any sequence $\{x^k\}$ satisfying $(1 - Q)\, x^{k-1} \in A^k(x^k)$ converges to solution of the QLCP (q, M).

Proof can be given in a line of the arguments given in Pang[5].

6. Concluding Remarks

As one of the concluding remarks we would like to point out that in the case of quasi-linear complementarity problems the algorithm which was developed in section 3, for the iterative solution technique can as well be generalized for variable parameters such as the case when the assumptions $E = E^k$ and $K = K^k$ are relaxed and we take uniformly bounded (by matrix norm) matrices E^k and K^k in the iterative process of the algorithm itself. The variable parameter algorithms are still possible to find the fixed point for the set-valued maps $A\,(x^k)$.

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