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ON SOME NEW DISCRETE INEQUALITIES IN TWO INDEPENDENT VARIABLES

B. G. PACHPATTE

Department of Mathematics and Statistics, Marathwada University Aurangabad 431004, (*Maharashtra*)

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The aim of this paper is to establish some new discrete inequalities in two independent variables which can be used as handy tools in the qualitative analysis of a new class of finite difference equations involving two independent variables.

1. INTRODUCTION

The fundamental role played by the discrete inequalities in the development of the theory of finite difference equations and numerical analysis is well known. A large number of papers dealing with discrete inequalities and their applications have appeared during the last few years, see[1-10] and some of the references given therein. Although stimulating research works have been undertaken in this direction, there are still a number of interesting classes of multidimensional finite difference equations which needs new types of discrete inequalities in their analysis. Our objective here is to present some new discrete inequalities in two independent variables which can be used as handy tools in the qualitative analysis of a new class of finite difference equations in two independent variables. In order to convey the importance of our results to the literature, we present applications of some of our inequalities to the study of boundedness, uniqueness and continuous dependence of the solutions of a new class of fourth order finite difference equations in two independent variables.

2. STATEMENT OF RESULTS

We first summarise some basic notations and definitions which will be used throughout this paper. Let $N_0 = \{0, 1, 2, \ldots\}$. The expression $u(0) + \sum_{s=0}^{n-1} b(s)$ represents a solution of the linear difference equation $\Delta u(n) = b(n)$ for $n \in N_0$, where Δ is the operator defined by $\Delta u(n) = u(n+1) - u(n)$. The expression $u(0) \prod_{s=0}^{n-1} b(s)$ represents a solution of the linear difference equation $u(n+1) = b(n)\, u(n)$ for $n \in N_0$. We use the usual convention of writing $\sum_{s \in \Phi} b(s) = 0$

and $\prod_{s \in \Phi} b(s) = 1$, if Φ is the empty set. We also use the the following notations of the operators

$$\Delta_1 u(m, n) = u(m+1, n) - u(m, n),$$

$$\Delta_2 u(m, n) = u(m, n+1) - u(m, n)$$

for $m, n \in N_0$. We often use the letters m and n to denote the two independent variables which are members of N_0.

For convenience we list the following hypotheses:

(H_1) $u(m, n)$ and $h(m, n)$ are real-valued nonnegative functions defined for $m, n \in N_0$.

(H_2) $p_1(m, n)$, $p_2(m, n)$, $p_3(m, n)$ are real-valued positive functions defined for $m, n \in N_0$.

(H_3) $a(m, n)$ is real-valued, positive and nondecreasing function in both the variables m and n in N_0.

(H_4) $u(m, n) \geqslant u_0 \geqslant 0$, u_0 is a constant, $h(m, n) > 0$ are real-valued functions defined for $m, n \in N_0$.

(H_5) $g(u)$ is continuous, nondecreasing real-valued function defined on an interval $I = [u_0, \infty)$, $u_0 \geqslant 0$ is a constant, and $g(u) > 0$ on (u_0, ∞), $g(u_0) = 0$.

(H_6) $q_1(m, n)$, $q_2(m, n)$, $q_3(m, n)$ are real-valued positive functions defined for $m, n \in N_0$.

(H_7) $W(u)$ is continuous, nondecreasing and submultiplicative real-valued function defined on an interval I, and $W(u) > 0$ on (u_0, ∞), $W(u_0) = 0$.

A useful two independent variable discrete inequality is embodied in the following theorem.

Theorem 1—Suppose (H_1) and (H_2) are true. If

$$u(m, n) \leqslant c + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \times \sum_{t=0}^{y-1} h(s, t)\, u(s, t) \qquad \ldots(1)$$

for $m, n \in N_0$, where c is a nonnegative constant, then

$$u(m, n) \leqslant c \prod_{x=0}^{m-1} \left[1 + \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \right.$$

$$\times \sum_{t=0}^{y-1} h(s, t) \Big] \qquad \ldots(2)$$

for $m, n \in N_0$.

A slightly different version of Theorem 1 is given in the following theorem.

Theorem 2—Suppose (H_1), (H_2) and (H_3) are true. If

$$u(m, n) \leqslant a(m, n) + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \times \sum_{t=0}^{y-1} h(s, t)\, u(s, t) \qquad \ldots(3)$$

for $m, n \in N_0$, then

$$u(m, n) \leqslant a(m, n) \prod_{x=0}^{m-1} \Big[1 + \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \times \sum_{t=0}^{y-1} h(s, t) \Big] \qquad \ldots(4)$$

for $m, n \in N_0$.

Another interesting and useful discrete inequality is established in the following theorem.

Theorem 3—Suppose (H_2), (H_4) and (H_5) are true. If

$$u(m, n) \leqslant c + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \times \sum_{t=0}^{y-1} h(s, t)\, g(u(s, t)) \qquad \ldots(5)$$

for $m, n \in N_0$, where c is a nonnegative constant, then for $0 \leqslant m \leqslant m_1$, $0 \leqslant n \leqslant n_1$, $m, m_1, n, n_1 \in N_0$,

$$u(m, n) \leqslant G^{-1} \Big[G(c) + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \times \sum_{t=0}^{y-1} h(s, t) \Big] \qquad \ldots(6)$$

where

$$G(r) = \int_{r_0}^{r} \frac{dy}{g(y)}, \; r \geqslant u_0 \text{ with } r_0 > u_0 \qquad \ldots(7)$$

G^{-1} is the inverse of G and $m_1, n_1 \in N_0$ are chosen so that

$$G(c) + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \sum_{t=0}^{y-1} \times h(s, t) \in \text{Dom}(G^{-1})$$

for $m, n \in N_0$ and $0 \leqslant m \leqslant m_1$, $0 \leqslant n \leqslant n_1$.

We next establish the following more general inequality which may be convenient in some applications.

Theorem 4—Suppose (H_1), (H_2), (H_6) and (H_7) are true. If

$$\begin{aligned} u(m, n) \leqslant c &+ \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \\ &\times \sum_{t=0}^{y-1} h(s, t)\, u(s, t) \\ &+ \sum_{x=0}^{m-1} \frac{1}{q_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{q_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{q_3(s, y)} \\ &\times \sum_{t=0}^{y-1} k(s, t)\, W(u(s, t)) \end{aligned} \qquad \ldots(8)$$

for $m, n \in N_0$, where c is a nonnegative constant and $k(m, n)$ is a real-valued nonnegative function defined for $m, n \in N_0$, then for $0 \leqslant m \leqslant m_2$, $0 \leqslant n \leqslant n_2$, $m, m_2, n, n_2 \in N_0$

$$\begin{aligned} u(m, n) \leqslant Q(m, n)\, \Omega^{-1}\Bigg[\Omega(c) &+ \sum_{x=0}^{m-1} \frac{1}{q_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{q_2(s, n)} \\ &\times \sum_{y=0}^{n-1} \frac{1}{q_3(s, y)} \sum_{t=0}^{y-1} k(s, t)\, W(Q(s, t)) \Bigg] \end{aligned} \qquad \ldots(9)$$

where

$$Q(m, n) = \prod_{x=0}^{m-1} \Big[1 + \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{p}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t) \Big] \qquad \ldots(10)$$

and

$$\Omega(r) = \int_{r_0}^{r} \frac{dy}{W(y)}, \quad r \geq u_0 \text{ with } r_0 > u_0 \qquad \ldots(11)$$

Ω^{-1} is the inverse of Ω and $m_2, n_2 \in N_0$ are chosen so that

$$\Omega(c) + \sum_{x=0}^{m-1} \frac{1}{q_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{q_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{q_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} k(t, s) W(Q(t, s)) \in \mathrm{Dom}(\Omega^{-1})$$

for $m, n \in N_0$ and $0 \leqslant m \leqslant m_2$, $0 \leqslant n \leqslant n_2$.

3. Proofs of Theorems 1—4

In order to establish the inequality (2) in Theorem 1, we first assume that $c > 0$ and define a function $z(m, n)$ by

$$z(m, n) = c + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t) u(s, t). \qquad \ldots(12)$$

From (12) it is easy to observe that

$$z(0, n) = z(m, 0) = c \qquad \ldots(13)$$

and

$$p_1(m, n) \Delta_1 z(m, n) = \sum_{s=0}^{m-1} \frac{1}{p_2(s, y)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, n)} \sum_{t=0}^{y-1} h(s, t) u(s, t)$$

$$\ldots(14)$$

$$p_2(m, n)\, \Delta_1 [p_1(m, n)\, \Delta_1 z(m, n) = \sum_{y=0}^{n-1} \frac{1}{p_3(m, y)} \sum_{t=0}^{y-1} h(m, t)\, u(m, t) \quad \ldots(15)$$

$$p_3(m, n)\, \Delta_2 [p_2(m, n)\, \Delta_1 [p_1(m, n)\, \Delta_1 z(m, n)]] = \sum_{t=0}^{n-1} h(m, t)\, u(m, t) \quad \ldots(16)$$

$$\Delta_2 [p_3(m, n)\, \Delta_2 [p_2(m, n)\, \Delta_1 [p_1(m, n)\, \Delta_1 z(m, n)]]] = h(m, n)\, u(m, n). \quad \ldots(17)$$

Using the fact that $(u, m, n) \leqslant z(m, n)$ in (17) we have

$$\Delta_2 (p_3(m, n)\, \Delta_2 [p_2(m, n)\, \Delta_1 [p_1(m, n)\, \Delta_1 z(m, n)]]] \leqslant h(m, n)\, z(m, n). \quad \ldots(18)$$

From the definition of $u(m, n)$ we observe that $z(m, n) \leqslant z(m, n+1)$ for $m, n \in N_0$. Using this fact in (18) we see that

$$\frac{p_3(m, n+1)\, \Delta_2 [p_2(m, n+1)\, \Delta_1 [p_1(m, n+1)\, \Delta_1 z(m, n+1)]]}{z(m, n+1)} - \frac{p_3(m, n)\, \Delta_2 [p_2(m, n)\, \Delta_1 [p_1(m, n)\, \Delta_1 z(m, n)]]}{z(m, n+1)} \leqslant h(m, n). \quad \ldots(19)$$

From (19) and the fact that $p_3(m, n)\, \Delta_2 [p_2(m, n)\, \Delta_1 [p_1(m, n)\, \Delta_1 z(m, n)]] \geqslant 0$ from (16), we observe that

$$\frac{p_3(m, n+1)\, \Delta_2 [p_2(m, n+1)\, \Delta_1 [p_1(m, n+1)\, \Delta_1 z(m, n+1)]]}{z(m, n+1)} - \frac{p_3(m, n)\, \Delta_2 [p_2(m, n)\, \Delta_1 [p_1(m, n)\, \Delta_1 z(m, n)]]}{z(m, n)} \leqslant h(m, n). \quad \ldots(20)$$

Now keeping m fixed in (20), set $n = t$ and sum over $t = 0, 1, 2, \ldots, n-1$ and use the fact that $p_3(m, 0)\, \Delta_2 [p_2(m, 0)\, \Delta_1 [p_1(m, 0)\, \Delta_1 z(m, 0)]] = 0$, from (16), to obtain the estimate

$$\frac{p_3(m, n)\, \Delta_2 [p_2(m, n)\, \Delta_1 [p_1(m, n)\, \Delta_1 z(m, n)]]}{z(m, n)} \leqslant \sum_{t=0}^{n-1} h(m, t). \quad \ldots(21)$$

From (21) and in view of the facts that $z(m, n) \leqslant z(m, n+1)$ and $p_2(m, n)\, \Delta_1 [p_1(m, n)\, \Delta_1 z(m, n)] \geqslant 0$, we observe that

$$\frac{p_2(m, n+1)\, \Delta_1 [p_1(m, n+1)\, \Delta_1 z(m, n+1)]}{z(m, n+1)}$$

$$- \frac{p_2(m, n) \Delta_1 [p_1(m, n) \Delta_1 z(m, n)]}{z(m, n)}$$

$$\leqslant \frac{1}{p_3(m, n)} \sum_{t=0}^{n-1} h(m, t). \qquad \ldots(22)$$

Keeping m fixed in (22) set $n = y$ and sum over $y = 0, 1, 2, \ldots, n-1$ and use the fact that $p_2(m, 0) \Delta_1 [p_1(m, 0) \Delta_1 z(m, 0)] = 0$ from (15), to obtain the estimate

$$\frac{p_2(m, n) \Delta_1 [p_1(m, n) \Delta_1 z(m, n)}{z(m, n)} \leqslant \sum_{y=0}^{n-1} \frac{1}{p_3(m, y)} \sum_{t=0}^{y-1} h(m, t). \qquad \ldots(23)$$

From (23) and in view of the facts that $z(m, n) \leqslant z(m+1, n)$ and $p_1(m, n) \Delta_1 z(m, n) \geqslant 0$ from (14), we observe that

$$\frac{p_1(m+1, n) \Delta_1 z(m+1, n)}{z(m+1, n)} - \frac{p_1(m, n) \Delta_1 z(m, n)}{z(m, n)}$$

$$\leqslant \frac{1}{p_2(m, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(m, y)} \sum_{t=0}^{y-1} h(m, t). \qquad \ldots(24)$$

Now keeping n fixed in (24), set $m = s$ and sum over $s = 0, 1, 2, \ldots, m-1$ and use the fact that $p_1(0, n) \Delta_1 z(0, n) = 0$ from (14), to obtain the estimate

$$\frac{\Delta_1 z(m, n)}{z(m, n)} \leqslant \frac{1}{p_1(m, n)} \sum_{s=0}^{m-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \sum_{t=0}^{y-1} h(s, t). \qquad \ldots(25)$$

From (25) we see that

$$z(m+1, n) \leqslant z(m, n) \left[1 + \frac{1}{p_1(m, n)} \sum_{s=0}^{m-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \right.$$

$$\left. \times \sum_{t=0}^{y-1} h(s, t) \right]. \qquad \ldots(26)$$

Now keeping n fixed in (26), set $n = x$ and substitute $x = 0, 1, 2, \ldots, m-1$ successively and use the fact that $z(0, n) = c$ from (13), to obtain the estimate

$$z(m, n) \leqslant c \prod_{x=0}^{m-1} \left[1 + \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \right.$$

$$\times \sum_{t=0}^{y-1} h(s, t) \Big].$$

Substituting this bound on $z(m, n)$ on the right side of (1) we obtain the inequality in (2).

Now suppose $c = 0$. Then from (1) we see that the inequality

$$u(m, n) \leqslant \epsilon + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t)\, u(s, t)$$

holds for every arbitrary positive number ϵ and $m, n \in N_0$, which by the above argument yields the estimate

$$u(m, n) \leqslant \epsilon \prod_{x=0}^{m-1} \Big[1 + \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t) \Big]. \qquad \ldots(27)$$

Since $u(m, n) \geqslant 0$ and $\epsilon > 0$ is arbitrary number independent of m, n then from (27) it follows that $u(m, n) = 0$. This completes the proof of Theorem 1.

Since $a(m, n)$ is positive and nondecreasing, we observe from (3) that

$$\frac{u(m, n)}{a(m, n)} \leqslant 1 + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t) \frac{u(s, t)}{a(s, t)}.$$

Now an application of Theorem 1 yields the required bound in (4) and the proof of Theorem 2 is complete.

In order to establish the inequality (6) in Theorem 3, let $\epsilon > 0$ and $u_\epsilon(m, n) = u(m, n) + \epsilon \geqslant u_0$ for all $m, n \in N_0$. Then from (5) we see that

$$u_\epsilon(m,n) \leqslant c+\epsilon+\sum_{x=0}^{m-1}\frac{1}{p_1(x,n)}\sum_{s=0}^{x-1}\frac{1}{p_2(s,n)}\sum_{y=0}^{n-1}\frac{1}{p_3(s,y)}$$

$$\times \sum_{t=0}^{y-1} h(s,t)\,g(u_\epsilon(s,t)-\epsilon)$$

$$\leqslant c+\epsilon+\sum_{x=0}^{m-1}\frac{1}{p_1(x,n)}\sum_{s=0}^{x-1}\frac{1}{p_2(s,n)}\sum_{y=0}^{n-1}\frac{1}{p_3(s,y)}$$

$$\times \sum_{t=0}^{y-1} h(s,t)\,g(u_\epsilon(s,t)). \qquad ..(28)$$

Define a function $z(m,n)$ by

$$z(m,n) = c+\epsilon+\sum_{x=0}^{m-1}\frac{1}{p_1(x,n)}\sum_{s=0}^{x-1}\frac{1}{p_2(s,n)}\sum_{y=0}^{n-1}\frac{1}{p_3(s,y)}$$

$$\times \sum_{t=0}^{y-1} h(s,t)\,g(u_\epsilon(s,t)). \qquad ...(29)$$

From (29) it is easy to observe that

$$z(m,0) = z(0,n) = c+\epsilon \qquad ...(30)$$

and

$$p_1(m,n)\,\Delta_1 z(m,n) = \sum_{s=0}^{m-1}\frac{1}{p_2(s,n)}\sum_{y=0}^{n-1}\frac{1}{p_3(s,y)}\sum_{t=0}^{y-1} h(s,t)\,g(u_\epsilon(s,t)). \qquad ...(31)$$

$$p_2(m,n)\,\Delta_1[p_1(m,n)\,\Delta_1 z(m,n)] = \sum_{y=0}^{n-1}\frac{1}{p_3(m,y)}\sum_{t=0}^{y-1} h(m,t)\,g(u_\epsilon(m,t)) \qquad ...(32)$$

$$p_3(m,n)\,\Delta_2[p_2(m,n)\,\Delta_1[p_1(m,n)\,\Delta_1 z(m,n)]] = \sum_{t=0}^{n-1} h(m,t)\,g(u_\epsilon(m,t)) \qquad ...(33)$$

$$\Delta_2[p_3(m,n)\,\Delta_2[p_2(m,n)\,\Delta_1[p_1(m,n)\,\Delta_1 z(m,n)]]] = h(m,n)\,g(u_\epsilon(m,n)). \qquad ...(34)$$

Using the fact that $u_\epsilon(m, n) \leqslant z(m, n)$ in (34) we have

$$\Delta_2 [p_3(m, n) \Delta_2 [p_2(m, n) \Delta_1 [p_1(m, n) \Delta_1 z(m, n)]]] \leqslant h(m, n) g(z(m, n)). \quad \ldots(35)$$

From the definition of $z(m, n)$ in (29) we observe that $z(m, n) \leqslant z(m, n+1)$ for $m, n \in N_0$. Using this and the fact that

$$p_3(m, n) \Delta_2 [p_2(m, n) \Delta_1 [p_1(m, n) \Delta_1 z(m, n)]] \geqslant 0$$

from (33), we observe from (35) that

$$\frac{p_3(m, n+1) \Delta_2 [p_2(m, n+1) \Delta_1 [p_1(m, n+1) \Delta_1 z(m, n+1)]]}{g(z(m, n+1))} - \frac{p_3(m, n) \Delta_2 [p_2(m, n) \Delta_1 [p_1(m, n) \Delta_1 z(m, n)]]}{g(z(m, n))} \leqslant h(m, n). \quad \ldots(36)$$

Now by following exactly the same steps as in the proof of Theorem 1 below the inequality (20) up to the inequality (25) with suitable changes, we obtain

$$\frac{\Delta_1 z(m, n)}{g(z(m, n))} \leqslant \frac{1}{p_1(m, n)} \sum_{s=0}^{m-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \sum_{t=0}^{y-1} h(s, t). \quad \ldots(37)$$

From (7) and (37) we have

$$G(z(m+1, n)) - G(z(m, n)) = \int_{z(m,n)}^{z(m+1,n)} \frac{dy}{g(y)} \leqslant \frac{\Delta_1 z(m, n)}{g(z(m, n))}$$

$$\leqslant \frac{1}{p_1(m, n)} \sum_{s=0}^{m-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \times \sum_{t=0}^{y-1} h(s, t). \quad \ldots(38)$$

Now keeping n fixed in (38), set $m = x$ and sum over $x = 0, 1, 2, \ldots, m-1$ to obtain the estimate

$$G(z(m, n)) \leqslant G(c + \epsilon) + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{n-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \times \sum_{t=0}^{y-1} h(s, t). \quad \ldots(39)$$

The bound in (6) now follows by substituting the bound for $z(m, n)$ from (39) in (28) and letting $\epsilon \to 0$. The subintervals of N_0 for m, n are obvious and the proof of Theorem 3 is complete.

In order to prove the inequality (9) in Theorem 4, let $\epsilon > 0$ and $u_\epsilon(m, n) = u(m, n) + \epsilon \geqslant u_0$ for $m, n \in N_0$. Then from (8) we see that

$$u_\epsilon(m, n) \leqslant c + \epsilon + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t)(u_\epsilon(s, t) - \epsilon) + \sum_{x=0}^{m-1} \frac{1}{q_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{q_2(s, n)}$$

$$\times \sum_{y=0}^{n-1} \frac{1}{q_3(s, n)} \sum_{t=0}^{y-1} k(s, t) W(u_\epsilon(s, t) - \epsilon)$$

$$\leqslant c + \epsilon + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t) u_\epsilon(s, t) + \sum_{x=0}^{m-1} \frac{1}{q_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{q_2(s, n)}$$

$$\times \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)} \sum_{t=0}^{y-1} k(s, t) W(u_\epsilon(s, t)). \qquad \ldots(40)$$

Define

$$a(m, n) = c + \epsilon + \sum_{x=0}^{m-1} \frac{1}{q_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{q_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{q_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} k(s, t) W(u_\epsilon(s, t)) \qquad \ldots(41)$$

then (40) can be restated as

$$u_\epsilon(m, n) \leqslant a(m, n) + \sum_{x=0}^{m-1} \frac{1}{p_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} h(s, t) u_\epsilon(s, t).$$

Since $a(m, n)$ is positive and nondecreasing function in both the variables m and n, we have from Theorem 2

$$u_\varepsilon(m, n) \leqslant a(m, n)\, Q(m, n) \qquad \dots(42)$$

where $Q(m, n)$ is as defined in (10). Since W is submultiplicative, we have

$$W(u_\varepsilon(m, n)) \leqslant W(a(m, n))\, W(Q(m, n)). \qquad \dots(43)$$

From (41) and (43) we have

$$a(m, n) \leqslant c + \epsilon + \sum_{x=0}^{m-1} \frac{1}{q_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{q_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{q_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} k(s, t)\, W(Q(s, t))\, W(a(s, t)).$$

Now by following the proof of Theorem 3 with suitable modifications we obtain

$$a(m, n) \leqslant \Omega^{-1}\left[\Omega(c + \epsilon) + \sum_{x=0}^{m-1} \frac{1}{q_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{q_2(s, n)}\right.$$

$$\left. \times \sum_{y=0}^{n-1} \frac{1}{q_3(s, y)} \sum_{t=0}^{y-1} k(s, t)\; W(Q(s, t))\right]. \qquad \dots(44)$$

The desired bound in (9)now follows by substituting (44) in (42) and letting $\epsilon \to 0$. The subintervals of N_0 for m and n are obvious. This completes the proof of Theorem 4.

4. Some Applications

In this section, we present some applications of our results to the study of boundedness, uniqueness and continuous dependence of the solutions of a new class of nonlinear finite difference equations in two independent variables. Each of these applications could be stated formally as a theorem. This has not been done so as not to obscure the essential ideas with technical details.

Example 1—As a first application, we obtain a bound on the solution of a nonlinear fourth order finite difference equation

$$\Delta_2\left[a_3(m, n)\, \Delta_2\left[a_2(m, n)\, \Delta_1\left[a_1(m, n)\; \Delta_1 u(m, n)\right]\right]\right] = f(m, n, u)(m, n)) \qquad \dots(45)$$

with the given boundary conditions at $m = 0,\ n = 0$

$$u(0, n) = \phi_1(n)$$

$$a_1(0, n)\, \Delta_1 u(0, n) = \phi_2(n)$$

$$a_2(m,0)\ \Delta_1[a_1(m,0)\ \Delta_1 u(m,0)] = \psi_1(m)$$

$$a_3(m,0)\ \Delta_2[a_2(m,0)\ \Delta_1[a_1(m,0)\ \Delta_1 u(m,0)]] = \psi_2(m). \qquad \text{...(46)}$$

Here a_1, a_2, a_3 are real valued positive functions defined on N_0^2, $f: N_0^2 \times R \to R$, where R denotes the set of real numbers; $\phi_1(n)$, $\phi_2(n)$, $\psi_1(m)$, $\psi_2(m)$ are real-valued nonnegative functions defined for $m, n \in N_0$. We assume that

$$|f(m,n,u)| \leqslant h(m,n)\,|u| \qquad \text{...(47)}$$

where $h(m,n)$ is a real-valued nonnegative function defined for $m, n \in N_0$. It is easy to observe that the problem (45) – (46) is equivalent to the equation

$$u(m,n) = b(m,n) + \sum_{x=0}^{m-1} \frac{1}{a_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{a_3(s,y)}$$

$$\times \sum_{t=0}^{y-1} f(s,t,u(s,t)) \qquad \text{...(48)}$$

where

$$b(m,n) = \phi_1(n) + \sum_{x=0}^{m-1} \frac{1}{a_1(x,n)} \phi_2(n) + \sum_{x=0}^{m-1} \frac{1}{a_1(x,n)}$$

$$\sum_{s=0}^{x-1} \frac{1}{a_2(s,n)} \psi_1(s) + \sum_{x=0}^{m-1} \frac{1}{a_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{a_2(x,n)}$$

$$\times\ \psi_2(s) \sum_{y=0}^{n-1} \frac{1}{a_3(s,y)}. \qquad \text{...(49)}$$

Suppose that

$$|b(m,n)| \leqslant k \qquad \text{...(50)}$$

where k is a nonnegative constant. Using (47), (50) in (48) we have

$$|u(m,n)| \leqslant k + \sum_{x=0}^{m-1} \frac{1}{a_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{a_3(s,y)}$$

$$\times \sum_{t=0}^{y-1} h(s,t)\,|u(s,t)|.$$

Now an application of Theorem 1 yields the bound on the solution $u(m,n)$ of (45)-(46) in terms of the known functions.

Example 2—As a second application, we shall discuss the uniqueness of the solution of the problem (45) – (46). We assume that the function f in (45) satisfies

$$| f(m, n, u) - f(m, n, \bar{u}) | \leqslant h(m, n) | u - \bar{u} | \quad \text{...(51)}$$

where $h(m, n)$ is as in Example 1. The problem (45)—(46) is equivalent to the equation (48). Then for any two solutions u and $\bar{u}$ of (45)—(46) we have

$$| u(m, n) - \bar{u}(m, n) | \leqslant \epsilon + \sum_{x=0}^{m-1} \frac{1}{a_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s, n)}$$

$$\times \sum_{y=0}^{n-1} \frac{1}{a_3(s, y)} \sum_{t=0}^{y-1} h(s, t) | u(s, t) - \bar{u}(s, t) | \quad \text{...(52)}$$

where $\epsilon > 0$ is arbitrary constant. The assumption (51) is used to get the inequality in (52). Now an application of Theorem 1 yields

$$| u(m, n) - \bar{u}(m, n) |$$

$$\leqslant \epsilon \left\{ \prod_{x=0}^{n-1} \left[1 + \frac{1}{a_1(x, n)} \sum_{s=u}^{x-1} \frac{1}{a_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{a_3(s, y)} \sum_{t=0}^{y-1} h(s, t) \right] \right\}.$$

Since $\epsilon > 0$ is arbitrary we have $u = \bar{u}$ i. e. there is at most one solution of the problem (45) – (46).

Example 3—Our third application is an example of continuous dependence of the solution on the equation and boundary data. Consider the problem (45)—(46) in Example 1 and the problem

$$\Delta_2 [a_3(m, n) \Delta_2 [a_2(m, n) \Delta_1 [a_1(m, n) \Delta_1 z(m,n)]]] = F(m, n, z(m, n)) \quad \text{...(53)}$$

with the given boundary conditions at $m = 0, n = 0$

$$z(0, n) = \bar{\phi}_2(n)$$

$$a_1(0, n) \Delta_1 z(0, n) = \bar{\psi}_2(n)$$

$$a_2(m, 0) \Delta_1 [a_1(m, 0) \Delta_1 z(m, 0)]] = \bar{\psi}_1(m)$$

$$a_3(m, 0) \Delta_2 [a_2(m, 0) \Delta_1 [a_1(m, 0) \Delta_1 z(m, 0)]] = \bar{\psi}_2(m). \quad \text{...(54)}$$

Here a_1, a_2, a_3 are as in Example 1, $F : N_0^2 \times R \to R$, $\bar{\phi}_1(n)$, $\bar{\phi}_2(n)$, $\bar{\psi}_1(m)$, $\bar{\psi}_2(m)$ are real-valued nonnegative functions defined for $m, n \in N_0$. The equations equivalent to (45) – (46) and (53) – (54) are (48) and

$$z(m, n) = \bar{b}(m, n) + \sum_{x=0}^{m-1} \frac{1}{a_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s, n)} \sum_{y=0}^{n-1} \frac{1}{a_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} F(s, t, z(s, t)) \qquad \ldots(55)$$

where $\bar{b}(m, n)$ is obtained from the definition of $b(m, n)$ by replacing $\phi_1(n)$, $\phi_2(n)$, $\psi_1(m)$, $\psi_2(m)$ in the right side in (49) by $\bar{\phi}_1(n)$, $\bar{\phi}_2(n)$, $\bar{\psi}_1(m)$, $\bar{\psi}_2(m)$ respectively. From (48) and (55) we have

$$u(m, n) - z(m, n)$$

$$= b(m, n) - \bar{b}(m, n)$$

$$+ \sum_{x=0}^{m-1} \frac{1}{a_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s, n)} \sum_{y=0}^{y-1} \frac{1}{a_3(s, y)}$$

$$\times \sum_{t=0}^{y-1} \{f(s, t, u(s, t)) - F(s, t, z(s, t)\}. \qquad \ldots(56)$$

Suppose that the function f in (45) satisfies the condition (51) and further we assume that

$$| b(m, n) - \bar{b}(m, n) | \leqslant \epsilon \qquad \ldots(57)$$

$$\sum_{x=0}^{m-1} \frac{1}{a_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s, n)} \sum_{y=0}^{x-1} \frac{1}{a_3(s, y)} \sum_{t=0}^{y-1} | f(s, t, z(s, t))$$

$$- F(s, t, z)(s, t)) | \leqslant \epsilon \qquad \ldots(58)$$

where $\epsilon > 0$ is arbitrary constant. By substracting and adding $f(s, t, z(s, t))$ in the braces on the right side of equation (56) and using (51), (57), (58), we obtain

$$| u(m, n) - z(m, n) | \leqslant 2\epsilon + \sum_{x=0}^{m-1} \frac{1}{a_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s, n)}$$

$$\sum_{y=0}^{n-1} \frac{1}{a_3(s, y)} \sum_{t=0}^{y-1} h(s, t) \, | u(s, t) - z(s, t) | . \qquad \ldots(59)$$

Now an application of Theorem 1 yields

$$| u(m, n) - z(m, n) |$$

$$\leq 2\epsilon \left\{ \prod_{x=0}^{m-1} \left[1 + \frac{1}{a_1(x, n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s, n)} \sum_{v=0}^{n-1} \frac{1}{a_3(s, y)} \right.\right.$$

$$\left.\left. \sum_{t=0}^{y-1} h(s, t) \right] \right\} . \qquad \ldots(60)$$

If $h(m, n)$ is bounded on some compact set $0 \leq m \leq m_0$, $0 \leq n \leq n_0$, $m, m_0, n, n_0 \in N_0$, then the quantity in braces on the right in (60) is bounded by some constant M on the set $0 \leq m \leq m_0$, $0 \leq n \leq n_0$. Therefore $|u(m, n) - z(m, n)| \leq 2M\epsilon$ on the set $0 \leq m \leq m_0$, $0 \leq n \leq n_0$; so the solution $u(m, n)$ of (45) – (46) depends continuously on f and the boundary data. If $\epsilon \to 0$, then $|u(m, n) - z(m, n)| - 0$ on this set.

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